Oral Exam Questions
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Functional Analysis

[1] Give a lecture on Hahn-Banach theorems.

[2] Consider the special case of Hilbert spaces. Can the Hahn-Banach theorem be proved here without Zorn's lemma? How? What property of the norm characterizes Hilbert spaces amongst Banach spaces? What Hilbert-like feature do uniformly convex spaces have?

[3] Define Banach algebras, and give some interesting examples. Does convolution and pointwise addition of functions in $L_1(\mathbb{R})$ make it a Banach algebra? Does the algebra have a unit? Let $\varphi_f : L_1 \to L_1$ be left convolution by f. Does the Banach algebra $\{\varphi_f\}_{\{f \in L_1\}}$ have a representation on $C(X)$, or a subset of $C(X)$, for some topological space X? When a formal unit is adjoined to L_1, what is X? Give explicitly the $*$-isomorphism. What is the spectrum of the operator φ_f in terms of f? Are there any invertible elements in this algebra? Describe the idempotents. In general, what topological property of X allows the existence of non-trivial idempotents in $C(X)$?

[4] Let $A : V \to W$ be a linear map of finite-dimensional vector spaces. Formulate conditions on the solvability of the equation $Ax = y$ in terms of the kernel of a certain operator. For what large class of maps A do these conditions generalise well when V, W are infinite-dimensional Hilbert spaces? If $A = I + K$, where K is compact and injective, find conditions of solvability when y is restricted to a subspace of finite co-dimension (do it first when K is self-adjoint).

Probability

[1] Given a measurable space $(S, \mathcal{S})$, and a σ-finite measure μ on this space, what are the axioms defining the Poisson process on S of intensity μ? Proof or counterexample: the third axiom making the process a random measure is not required for uniqueness.

[2] Take $S = \mathbb{R}$, and μ_k have density, with respect to Lebesgue measure, proportional to $x^{-k/2}$, for $k = 1, 3, 5$. Which ones define σ-finite measures? For each Poisson process, which subsets of $\mathbb{R}$ do you expect to contain a finite or infinite number of points? For which measures can you define the "sum of the points" of the process?

[3] If X_i are the decimal digits of an randomly chosen real number, and $S_n = \sum_{k=1}^{n} X_k$. Given $b > 4.5$, find an upper bound on $P(S_n/n > b)$ of form $(\varphi(\theta))^n$, $\theta \geq 0$. Show that for some value θ_0, the bound is non-trivial, i.e. $\varphi(\theta_0) < 1$.