Oral Exam Questions - Timothy DeVries (2007)

Enumerative Combinatorics

1. Let $\mathcal{W}$ be the class of binary words over the alphabet $\mathcal{A} = \{a, b\}$. For $w \in \mathcal{W}$, define a *rise* in w to be an occurrences of the string ab as a contiguous subword. Find the ordinary generating function $F(z, u)$ for $\mathcal{W}$, where z marks length and u marks number of rises. How would you calculate the expected number of rises in a randomly chosen word of length n? Without performing the calculation, what do you expect the answer to be?

2. For a fixed r, find the exponential generating function $F(z, u)$ for the class of permutations where z marks size and u marks number of r-cycles. Calculate the expected number of r-cycles in a random permutation of size n. Use this to calculate the expected number of cycles in a random permutation of size n.

3. What is a Catalan tree? Find the ordinary generating function $F(z, u)$ for the class of Catalan trees where z marks size and u marks the number of leaves. Find the expected number of leaves in a random Catalan tree of size n.

Asymptotics

1. Given the recurrence $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$ with $a_0 = 1$ and $a_j = 0$ for j negative, how would you determine the value of a_n for arbitrary n? Given the bivariate recurrence $a_{n,k} = a_{n-1,k} + a_{n,k-1} + a_{n-1,k-1}$ ($a_{0,0} = 1$; $a_{i,j} = 0$ for $i < 0$ or $j < 0$), step through the process of estimating $a_{n,k}$ via bivariate asymptotic methods. Finally, describe the kernel method and explain how it is used to compute the generating function of a well-founded recurrence relation.

2. Let $F(z) = \sum_{n=0}^{\infty} f_n z^n$ be the generating function defined by $F(z) = (1-z)^{-1/3} e^{-z^2/2}$. In as much specificity as you like, state the theorems that you need to analyze f_n asymptotically. Use the theorems you stated to find an asymptotic expression for f_n. Sketch a proof of one of the theorems, e.g. the big-Oh transfer theorem for $g(z) = O((1-z)^{-\alpha})$.

3. As an extension of the first enumerative combinatorics question, let w_n be the number of words of length n with $n/3$ rises. Using bivariate asymptotic methods, compute the exponential growth of w_n as n goes to infinity.

Logic and Finite Model Theory

1. Over the language of undirected simple graphs, let φ be any first order sentence. Let P_n be the probability that a random such graph on n vertices is a model of φ. Show that there exists a decision procedure to determine whether or not P_n goes to 1 as n goes to infinity.

2. Let G be an acyclic, 2-regular, undirected simple graph. Let φ be any first order sentence in the language consisting of one binary predicate (interpreted as edge relation). Show that if φ is true of G, then φ is true of some finite structure A. Use your proof to show that the property of being acyclic can not be captured in first order logic.