OVERTWISTED ENERGY-MINIMIZING CURL EIGENFIELDS

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ABSTRACT. We consider energy-minimizing divergence-free eigenfields of the curl operator in dimension three from the perspective of contact topology. We give a negative answer to a question of Etnyre and the first author by constructing curl eigenfields which minimize L^2 energy on their co-adjoint orbit, yet are orthogonal to an OVERTWISTED contact structure. We conjecture that K-contact structures on S^1-bundles always define TIGHT minimizers, and prove a partial result in this direction.

1. INTRODUCTION

Eigenfields of the curl operator form an important class of solutions to the steady Euler equations in dimension three. These equations model the velocity field of an inviscid, incompressible fluid flow on a Riemannian manifold [1]. It has been observed [6, 5, 8] that there is a correspondence between curl eigenfields and contact 1-forms in dimension three. Recall that a contact 1-form $\alpha \in \Omega^1(M^3)$ is one which satisfies $\alpha \wedge d\alpha \neq 0$. Such 1-forms have as their kernel a totally nonintegrable plane field known as a *contact structure*. The correspondence is this: any nonvanishing curl eigenfield is dual to a contact 1-form via the Riemannian metric; and any contact 1-form can be realized as dual to a nonvanishing curl eigenfield for some Riemannian structure.

This observation raises interesting questions concerning the interplay between fluid dynamical properties of curl eigenfields and topological properties of contact structures. More specifically, one can investigate how the topological tight/overtwisted dichotomy for contact structures relates to the physical properties of the fluid like energy, periodic orbits etc. In [8] (p. 16) the authors asked if variational principles of the fluid flows descend to variational principles for the corresponding contact structures.

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Question 1.1. Does a nonvanishing curl eigenfield which minimizes (L^2) energy on its coadjoint orbit under the volume-preserving diffeomorphism group necessarily define a tight contact structure?

Using a recent result of the second author [10], we give a negative answer to this question. This involves constructing special S^1-invariant curl eigenfields on products $S^1 \times \Sigma$, where Σ is a closed orientable surface of genus $g > 0$. In the second part of this note, we demonstrate that under additional symmetry conditions, the energy-minimization condition does yield a tightness constraint on the associated contact structures.

Throughout the paper, we use the language of differential forms and global analysis, as in [1]. We restrict attention to the class of volume-preserving vector fields and 1-forms exclusively.

2. CONTACT STRUCTURES

Let (M, g) be a Riemannian 3-manifold. The CURL OPERATOR on 1-forms is $*\,d : \Omega^1(M) \to \Omega^1(M)$, where $*$ is the Hodge star operator and d the exterior derivative. An EIGENFORM of curl is any 1-form α satisfying $*\,d\alpha = \mu\alpha$ for some $\mu \in \mathbb{R}$.

Given a nonvanishing curl eigenform α with nonzero eigenvalue $\mu \neq 0$, it easily follows that

$$\alpha \wedge d\alpha = \alpha \wedge (\mu * \alpha) \neq 0.$$

Therefore, α is a CONTACT FORM and the plane field $\xi = \ker(\alpha)$ defines a *contact structure* — a nowhere integrable plane field on M. If α is the dual 1-form to a vector field, then ξ is the orthogonal plane field. All contact structures in this paper are transversely orientable, as they are dual to globally-defined vector fields.

It has been known since the work of Bennequin and Eliashberg [2, 7] that there are two fundamentally different classes of contact structures.

Definition 2.1. *A contact structure ξ is* OVERTWISTED *if and only if there exists an embedded disk $D^2 \subset M$ such that D is transverse to ξ near ∂D but ∂D is tangent to ξ. Any contact structure which is not overtwisted is called* TIGHT.

As one might expect from the definition, it is rather difficult to determine if a given contact structure is tight or overtwisted. One of the more successful recent techniques for solving the classification problem involves examining the CHARACTERISTIC SURFACE.

Definition 2.2. *Let X be a vector field preserving the contact plane distribution ξ, i.e., $\mathcal{L}_X \xi = 0$. The characteristic surface Γ_X is the set of tangencies of X with ξ,*

$$\Gamma_X = \{p \in M : X_p \in \xi_p\}. \tag{1}$$

The following result is essential for our study of S^1-invariant curl eigenforms on circle bundles.

Theorem 2.3 (Giroux [9])**.** *Let ξ be an S^1-invariant contact structure on a principal circle bundle $\pi : P \to \Sigma$ over a closed oriented surface Σ. Let $\Gamma = \pi(\Gamma_{S^1})$ be a projection of the characteristic surface Γ_{S^1} onto Σ. Denote by $e(P)$ be the Euler number of the bundle P. Then ξ and all covers of ξ are tight if and only if one of the following holds:*

(i) *For $\Sigma \neq S^2$ none of the connected components of Σ/Γ is a disc.*
(ii) *For $\Sigma = S^2$, $e(P) < 0$ and $\Gamma = \emptyset$.*
(iii) *For $\Sigma = S^2$, $e(P) \geq 0$ and Γ is connected.*

A contact structure all of whose covers are tight is called UNIVERSALLY TIGHT.

3. ENERGY AND EIGENVALUES

An important feature of any curl eigenform α is the fact that it extremizes the L^2-energy, defined as,

$$E(\alpha) = \|\alpha\|^2_{L^2} = \int_M \alpha \wedge *\alpha, \tag{2}$$

among all 1-forms obtained from α by pullbacks through volume preserving diffeomorphisms. The set of such forms, Ξ_α, is the COADJOINT ORBIT of α under the action of the volume-preserving diffeomorphism group of M:

$$\Xi_\alpha = \{\beta : \beta = \varphi_*(\alpha),\ \varphi \in \mathrm{Diff}_0(M),\ \varphi_*(*1) = *1\}. \tag{3}$$

The question of energy minimization on the coadjoint orbit is more delicate, and closely related to spectral data. The following result is one of the few general results available:

[1]:

Proposition 3.1. *A curl eigenform α_1, (i.e. an eigenform of the curl operator $*\,d : \Omega^1(M) \to \Omega^1(M)$) corresponding to the first eigenvalue $\mu_1 \neq 0$ is a minimizer of the energy E on Ξ_{α_1}.*

Proof. The operator $*\,d$ is unbounded, closed, self-adjoint, and elliptic; it has a compact inverse $*\,d^{-1}$ defined on the orthogonal complement of its kernel (see [14]). We can also choose an orthonormal basis of eigenforms $\{\alpha_i\}$ in $(L^2(M,\Lambda^1 T^*M),(\cdot,\cdot)_{L^2})$ such that,

$$*\,d^{-1}\alpha_i = \frac{1}{\mu_i}\,\alpha_i, \qquad 0<\mu_1^2\leq\mu_2^2\leq\ldots\leq\mu_i^2\leq\ldots \tag{4}$$

For an arbitrary L^2 1-form $\beta\in\mathrm{Im}(\delta)$ we have

$$|(*\,d^{-1}\beta,\beta)_{L^2}| = |\sum_i \frac{1}{\mu_i}(\alpha_i,\beta)^2_{L^2}| \leq \frac{1}{|\mu_1|}(\beta,\beta)_{L^2} = \frac{1}{|\mu_1|}E(\beta). \tag{5}$$

One obtains a lower bound for the energy $E(\beta)$,

$$E(\beta)\geq|\mu_1||(*\,d^{-1}\beta,\beta)_{L^2}|.$$

The above inequality becomes the equality if and only if $\beta=\alpha_1$. The claim follows from the fact that the HELICITY, $(*\,d^{-1}\beta,\beta)_{L^2}$, is invariant under volume preserving transformations see [1]. □

Since the curl operator is a formally self-adjoint first-order elliptic operator of the first order, the principal eigenvalue μ_1 enjoys the variational characterization through the Rellich's quotient. As a corollary of the following considerations and Lemma 3.2 we have,

$$\mu_1=\inf_{\alpha\in\mathcal{H}}\frac{|(*\,d\alpha,\alpha)_{L^2}|}{\|\alpha\|^2_{L^2}} \quad\Rightarrow\quad \mu_1^2=\inf_{\alpha\in\mathcal{H}}\frac{(\Delta^1_M\alpha,\alpha)_{L^2}}{\|\alpha\|^2_{L^2}},$$
$$\mathcal{H}=\mathrm{Ker}(*\,d)^{\perp}=\{\alpha\in\Omega^1(M):\alpha=\delta\beta,\ \text{for some }\beta\in\Omega^2(M)\},$$

Indeed, it follows that the curl squared is equal to the Hodge Laplacian , $(*\,d)^2=\delta\,d$, on $\mathcal{H}$. Therefore any curl eigenform α, (i.e. $*\,d\alpha=\pm\mu\,\alpha$) is automatically a co-closed μ^2-eigenform of the Hodge Laplacian i.e.

$$\Delta^1_M\alpha=\delta\,d\,\alpha=*\,d*d\alpha=\mu^2\,\alpha, \tag{6}$$

Clearly the curl $*\,d$ commutes with the Hodge Laplacian Δ^1_M, therefore both of these operators are simultaneously diagonizable on $\mathcal{H}$ in a suitable orhonormal basis of curl eigenforms;

$$\mathcal{H}=\bigoplus_{i=1}^{\infty}E^{\Delta}(\mu_i^2),\quad E^{\Delta}(\mu_i^2)\perp E^{\Delta}(\mu_j^2),\ i\neq j \qquad 0<\mu_1^2\leq\mu_2^2\leq\ldots\leq\mu_i^2\leq\ldots$$

where $E^{\Delta}(\mu_i^2)$ stands for the μ_i^2-eigenspace of Δ^1_M, and

$$E^{\Delta}(\mu_i^2)=E^{*\,d}(\mu_i)\oplus E^{*\,d}(-\mu_i).$$

(We allow one of $E^{*\,d}(\mu_i)$, $E^{*\,d}(-\mu_i)$ to be trivial.) We may conclude further that there exist two positive operators $\sqrt{\Delta_+}$, $\sqrt{\Delta_-}$, such that $*\,d=\sqrt{\Delta_+}-\sqrt{\Delta_-}$.

The following useful fact, which can be traced back to work in [4], tells us how to effectively find a basis of curl eigenforms from a basis of co-closed Δ^1_M-eigenforms.

Lemma 3.2. *Any curl μ-eigenform is automatically a co-closed μ^2-eigenform of the Laplacian Δ^1_M. Conversely, given a co-closed μ^2-eigenform $\alpha \in \Omega^1(M)$ of Δ^1_M there exists a corresponding $\pm\mu$-curl eigenform $\beta \in \Omega^1(M)$ given by*

$$\beta = \mu\,\alpha \pm *\,d\alpha \tag{7}$$

Proof. We verify this claim by a direct calculation. Since $\delta\alpha = 0$, and $*d*d = \delta\, d = \Delta|_{\mathcal{H}}$, $\Delta^1_M\alpha = \mu^2\alpha$, $\delta\alpha = 0$ we obtain,

$$*\,d\beta = \mu * d\alpha \pm \Delta^1_M\alpha = \pm\mu^2 * \alpha + \mu * d\alpha = \pm\mu\,\beta$$

□

4. Overtwisted principal eigenfields

For the purpose of producing an overtwisted principal curl eigenform we assume that M is a trivial bundle $P = S^1 \times \Sigma$ and the metric g on P is a product metric $g = 1 \oplus g_\Sigma$ such that fibers are of constant length l. We will construct our example using a sequence of lemmas.

In the case of the product (P, g), $P = S^1 \times \Sigma$, $g = 1 \oplus g_\Sigma$, the space of smooth 1-forms $\Omega^1(P)$ decomposes with respect to the L^2-inner product induced by the metric g as,

$$\Omega^1(P) = \Omega^1_N(P) \oplus \Omega^1_T(P) \tag{8}$$

where,

$$\Omega^1_N(P) = \{\alpha \in \Omega^1(P) : \alpha = f\eta,\ f \in C^\infty(P)\},$$
$$\Omega^1_T(P) = \Omega^1_N(P)^\perp \cap \Omega^1(P),$$
$$\Omega^1_T(P) = \{\alpha \in \Omega^1(P) : \alpha(X_\eta) = 0\}$$

The following lemma is another consequence of the product metric assumption.

Lemma 4.1. *The Laplacian Δ^1_P preserves $\Omega^1_N(P)$, $\Omega^1_T(P)$ and for $\alpha = f\eta + \beta$, $f\eta \in \Omega^1_N(P)$, $\beta \in \Omega^1_T(P)$ we have the following formula for the Laplacian,*

$$\Delta^1_P\alpha = (\Delta^0_P f)\eta + \Delta^1_P\beta = (-\mathcal{L}^2_\eta + \Delta^0_\Sigma f)\eta + (-\mathcal{L}^2_\eta\,\beta + \Delta^1_\Sigma\,\beta) \tag{9}$$

Proof. The first claim immediately follows from the formula (9), and the formula itself from the general form of the product Laplacian on k-forms: $\Delta^k_{S^1\times\Sigma} = \sum_j \Delta^j_{S^1} \otimes \mathrm{Id} + \mathrm{Id} \otimes \Delta^{k-j}_\Sigma$ (see, e.g., [12]). □

Lemma 4.2. *On the product manifold $P = S^1 \times \Sigma$, $g = 1 \oplus g_\Sigma$ with constant length l fibres. The first eigenvalue μ_1 of the curl operator satisfies,*

$$\mu_1^2 = \min\left\{\nu_1, \Big(\frac{2\pi}{l}\Big)^2\right\}, \quad \textit{where} \quad \nu_1 = \inf_{f\in L^2(\Sigma),\ f\neq \textit{const}} \left\{\frac{(\Delta^0_\Sigma f, f)_{L^2}}{\|f\|^2_{L^2}}\right\}.$$

Proof. From the decomposition (8) and the fact that Δ^1_P preserves $\Omega^1_T(P)$ and $\Omega^1_N(P)$ (see Lemma 4.1) we have

$$\mu_1^2 = \min\left\{\mu^2_{1,T}, \mu^2_{1,N}\right\}; \quad \mu^2_{1,r} = \inf_{\alpha\in\mathcal{H}\cap\Omega^1_r(P)} \left\{\frac{(\Delta^1_P\alpha, \alpha)_{L^2}}{\|\alpha\|^2_{L^2}}\right\} \quad r = T, N. \tag{10}$$

In order to calculate $\mu^2_{1,N}$ notice that for any $\alpha \in \mathcal{H}\cap\Omega^1_N(P)$, $\alpha = f\,\eta$, the function f is constant on the fibres; hence $f \in C^\infty(\Sigma)$. Indeed, $\delta\alpha = 0$, and, since $\nabla\eta = 0$ in the adapted frame $X_\eta = \{X_1, X_2, X_3\}$, we obtain

$$0 = \delta\alpha = \iota(X_i)\nabla_i\alpha = \iota(X_i)(\nabla_i f\eta + f\nabla_i\eta) = \nabla_1 f = X_\eta f.$$

We conclude now that for any $\alpha \in \mathcal{H}\cap\Omega^1_N(P)$, $\Delta^1_\Sigma\alpha = (\Delta^0_\Sigma f)\eta$ and

$$(\Delta^1_P\alpha, \alpha)_{L^2} = (\Delta^0_\Sigma f\eta, \eta)_{L^2} = \int_P (f\,\Delta^0_\Sigma f)\eta\wedge *\eta = (\Delta^0_\Sigma f, f)_{L^2},$$

where $\eta\wedge *\eta = *1$. Therefore

$$\mu^2_{1,N} = \nu_1, \qquad \nu_1 = \inf_{f\in L^2(\Sigma),\ f\neq\text{const}} \left\{\frac{(\Delta^0_\Sigma f, f)_{L^2}}{\|f\|^2_{L^2}}\right\}. \tag{11}$$

In other words $\mu^2_{1,N}$ is equal to the first eigenvalue of the scalar Laplacian Δ^0_Σ on Σ.

In order to calculate $\mu^2_{1,T}$ we first calculate the orthogonal basis of eigenforms of $\mathcal{H}\cap\Omega^1_T(P)$. Let $\{\beta_m\}$ be an orthonormal basis of Δ^1_Σ- eigenforms on $\mathrm{Ker}(\Delta^1_\Sigma)\oplus\mathrm{Im}(\delta) \subset L^2(\Lambda^1T^*\Sigma)$, define for all $m, n \in \mathbb{Z}^+$:

$$h_0 = g_0 = 1, \quad h_n = \cos\Big(\frac{2\pi n t}{l}\Big),\ g_n = \sin\Big(\frac{2\pi n t}{l}\Big), \tag{12}$$

$$\alpha^g_{n\,m} = g_n\,\beta_m, \qquad \text{and} \qquad \alpha^h_{n\,m} = h_n\,\beta_m.$$

Clearly $\{\alpha^g_{n\,m}, \alpha^h_{n\,m}\}$ is a set of 1-eigenforms of Δ^1_P on $\mathcal{H}\cap\Omega^1_T(P)$. Indeed, from (9) we have,

$$\Delta^1_P\alpha^r_{mn} = \gamma^r_{mn}\alpha^r_{mn}, \quad \gamma^r_{mn} = \Big(\frac{2\pi n}{l}\Big)^2 + \tilde{\nu}_m$$

where $\tilde{\nu}_m$ is the m-th eigenvalue of Δ^1_Σ. One easily shows that $\{\alpha^g_{n\,m}, \alpha^h_{n\,m}\}$ is an orthonormal basis of $\mathcal{H}\cap\Omega^1_T(P)$. Consequently, all eigenforms of Δ^1_P on $\mathcal{H}\cap\Omega^1_T(P)$ are listed in (12), and we have

$$\mu^2_{1,T} = \min\left\{\Big(\frac{2\pi}{l}\Big)^2, \tilde{\nu}_1\right\} \tag{13}$$

It is left to show that $\tilde{\nu}_1 = \nu_1$. By the Hodge decomposition theorem:

$$\Omega^1(\Sigma) = \mathrm{Ker}(\Delta^1_\Sigma) \oplus \mathrm{Im}(d_\Sigma) \oplus \mathrm{Im}(\delta_\Sigma).$$

Moreover, $\Omega^0(\Sigma) \simeq \Omega^2(\Sigma)$ through the Hodge-star isometry. We conclude that $\mathrm{Im}(\delta_\Sigma) = \{* d_\Sigma f; f \in C^\infty(\Sigma)\}$. Since Δ^1_Σ commutes with $* d_\Sigma$, any ν_m-eigenfunction f_m results in a ν_m-eigenform $* d_\Sigma f_m$. Therefore $\tilde{\nu}_1 = \nu_1$ and the lemma follows from (10), (11), and (13). □

In [10] the following theorem was proved:

Theorem 4.3 ([10])**.** *For an arbitrary closed compact orientable surface Σ, there exists a smooth metric g_Σ such that a nodal set $f_1^{-1}(0)$ of the principal eigenfunction f_1 of Δ^0_Σ is a single embedded circle which bounds a disc in Σ.*

Combining the above theorem with Lemma 4.2 results in the following,

Theorem 4.4. *Let $\Sigma \neq S^2$ be an orientable surface of an arbitrary nonzero genus. One can prescribe a metric g_Σ on Σ such that there exist an overtwisted curl eigenfield v on the product manifold $(S^1 \times \Sigma, 1 \oplus g_\Sigma)$ which minimizes the energy* (2) *on the coadjoint orbit Ξ_α.*

Proof. In the first step we choose a metric g_Σ on $\Sigma \neq S^2$ constructed in Theorem 4.3 and assume that the length of fibres in $(S^1 \times \Sigma, 1 \oplus g_\Sigma)$ is equal to l. By Lemma 4.2 we may choose l small so that the first eigenvalue satisfies $\mu_1 = \nu_1$. By earlier considerations $E^\Delta(\mu_1^2) = E^{*\,d}(\mu_1) \oplus E^{*\,d}(-\mu_1)$ and $E^\Delta(\mu_1^2)$ is spanned by two independent $\pm\mu_1$- curl eigenforms defined by Lemma 3.2 as

$$\alpha_\pm = f_1\eta \pm *_\Sigma\, d\, f_1.$$

The dimension is two since g_Σ can be chosen in a residual subset of an open set of metrics: see, [10]. These forms are nonvanishing since the set of zeros is clearly equal to the singular part of the nodal set of f_1. This singular part is empty for a generic choice of metric (see [13]). Both forms are S^1-invariant and overtwisted by Theorem 2.3. Indeed, the projection of the characteristic surface Γ_{S^1} of $\alpha_\pm$ onto Σ is equal to $f_1^{-1}(0)$, the nodal set of f_1. By the choice of the metric $\pi(\Gamma_{S^1})$ bounds a disk. Now, the dual curl eigenfields $\alpha^\#_\pm$ minimize energy (2) on $\Xi_{\alpha_\pm}$ due to Proposition 3.1. □

5. Symmetry and tight energy minima

There are certain cases for which principal curl eigenforms can never be overtwisted. Courant's theorem on nodal sets quickly yields such a rigidity result for $S^2 \times S^1$:

Proposition 5.1. *Let $M = S^2 \times S^1$ with any product metric giving the S^1 fibers a constant length l. Then the principal eigenform of curl on M is never overtwisted.*

Proof. It follows from Lemma 4.2 that if the first eigenvalue μ_1 is equal to $\frac{2\pi}{l}$ then the curl eigenforms have zeros (since the dual vector fields are tangent to S^2). If $\mu_1 = \nu_1$ Theorem 2.3 tells us that a contact form is tight if and only if the projection of $\Gamma_{S^1} = f_1^{-1}(0)$ is a single circle. On the other hand Courant's theorem implies that the principal eigenfunction on a closed surface always has exactly two nodal domains, which in turn implies tightness. □

For the more general case, tightness can be forced by additional symmetry. Let P be a principal S^1-bundle over a closed orientable surface Σ, equipped with a bundle metric of constant length fibres. Assume that all fibers are geodesics, the bundle metric is invariant under the action of a Killing field X tangent to the fibres, and the projection $\pi : P \to \Sigma$ is a Riemannian submersion. Cartan's equations of structure imply that dual 1-form $\eta = X^\flat$ satisfies

$$* \, d\eta = 2\lambda \, \eta, \qquad \lambda \in C^\infty(P). \tag{14}$$

Therefore, η defines a curl eigenform, if $\lambda(x) = \lambda =$ const. It is shown in [11] that $\lambda = -e(P)/\gamma$, where $\gamma = \|X\|_p$ and $e(P)$ is an Euler number of the bundle. So called Sasakian structures are special case of such curl eigenforms and occur in the case $\lambda = 1$, Hopf fields on the round S^3 are the classical example (see [1]). Such structures are called K-CONTACT,[3], structures (see [11]). By Theorem 2.3 η is necessarily tight since the contact plane distribution is orthogonal to the fibers.

Denote by $\mathcal{H}_{S^1}$ the subspace of S^1- invariant 1-forms in $\mathcal{H} \subset \Omega^1(P)$.

Proposition 5.2. *Any curl eigenform η defined by a K-contact structure is always energy-minimizing on $\mathcal{H}_{S^1} \cap \Xi_\eta$. Let ν be the first nonzero eigenvalue of the scalar Laplacian on Σ. If $\nu > 3\lambda^2$ then η is a principal curl eigenform on $\mathcal{H}_{S^1}$.*

Proof. We provide the proof of the first claim for $\lambda > 0$ (in the case of $\lambda < 0$ the reasoning is analogous). The space $\mathcal{H}_{S^1}$ decomposes as

$$\mathcal{H}_{S^1} = \mathcal{H}^+_{S^1} \oplus \mathcal{H}^-_{S^1},$$

where $\mathcal{H}^\pm_{S^1}$ is a subspace spanned by positive/negative curl eigenforms. We need to show that η is an energy minimizer on $\mathcal{H}_{S^1} \cap \Xi_\eta$. Given a volume preserving diffeomorphism $\varphi : P \to P$ we denote $\eta_\varphi = \varphi_*(\eta) \in \Xi_\eta$. Under the assumptions on the φ action, $\eta_\varphi \in \mathcal{H}_{S^1} \cap \Xi_\eta$. We expand η_φ in the eigenbasis of curl eigenforms (4), $\eta_\varphi = \sum_{i \geq 0} c_i^+ \alpha_i^+ + \sum_{i<0} c_i^- \alpha_i^-$, where

$\{\alpha_i^\pm\}$ span $\mathcal{H}_{S^1}$. Since the helicity $(*d^{-1}\eta_\varphi, \eta_\varphi)$ is invariant under φ, as in (5), we obtain

$$0 < \frac{E(\eta)}{2\lambda} = (*d^{-1}\eta, \eta) = (*d^{-1}\eta_\varphi, \eta_\varphi) = \sum_{i\geq 0} \frac{(c^+)^2}{\mu_i^+} + \sum_{i<0} \frac{(c^-)^2}{\mu_i^-}$$

where $\mu_i^\pm$, positive/negative eigenvalues of $*d$ on $\mathcal{H}_{S^1}$. Since the second sum is negative we can estimate $\mu_1^+(*d^{-1}\eta, \eta) \leq E(\eta_\varphi)$. To finish the proof it suffices to show that $2\lambda = \mu_1^+$, then we obtain $E(\eta) \leq E(\eta_\varphi)$ which proves the claim.

We derive the equality $2\lambda = \mu_1^+$ by a calculation in the adapted orthonormal coframe $\{\eta^1 = \gamma\,\eta, \eta^2, \eta^3\}$ dual to $\{X_1 = \frac{1}{\gamma}X, X_2, X_3\}$, where X is the Killing field tangent to the fibers and $\ker\alpha_1 = \operatorname{span}\{X_2, X_3\}$. Let $\alpha_1 = a_i\eta^i = f\eta + \beta$ be the curl eigenform satisfying

$$*d\alpha_1 = \mu_1^+\alpha_1. \tag{15}$$

Since, $\mathcal{L}_X\alpha_1 = 0$, using the Cartan formula we obtain:

$$\begin{aligned} 0 &= \mathcal{L}_X\alpha = \mu_1^+\,\iota(X) * \alpha + df \\ -df = -\nabla_i f\,\eta^i &= \mu_1^+\,\gamma\,\iota(X_1) * \alpha \\ df = \nabla_1 f\,\eta^1 + \nabla_2 f\,\eta^2 + \nabla_3 f\,\eta^3 &= \mu_1^+\,\gamma\,(a_2\,\eta^3 - a_3\,\eta^2). \end{aligned}$$

This leads to the following equations,

$$\nabla_1 f = 0, \quad \nabla_2 f = -\mu_1^+\,\gamma\,a_3, \quad \nabla_3 f = \mu_1^+\,\gamma\,a_2. \tag{16}$$

Applying (15) and (16) to $d\,\alpha = \nabla_i a_k\eta^i \wedge \eta^k - a_k\Gamma^k_{ji}\eta^i \wedge \eta^j$ (where Γ^k_{ji} are the Christoffel symbols in the frame $\{\eta^i\}$) leads to,

$$\frac{\mu_1^+}{\gamma}\,f = \frac{1}{\mu_1^+\,\gamma}(-\nabla_{22}f - \nabla_{33}f + \Gamma^2_{33}\nabla_2 f + \Gamma^3_{22}\nabla_3 f) + \frac{2\lambda}{\gamma}f. \tag{17}$$

Consequently we obtain the following equation for f:

$$\Delta^0_P f = \mu_1^+(\mu_1^+ - 2\lambda)f. \tag{18}$$

Equations (16) imply that for α_1 to be nontrivial, f cannot be a constant zero function. Since Δ^0_P is a positive operator we conclude that $\mu_1^+ \geq 2\lambda$; consequently, $\mu_1^+ = 2\lambda$ because μ_1^+ is the first positive eigenvalue. The equation (18) is valid for any S^1-invariant μ-eigenform. Thus the proof of the second statement follows from the equation $\nu = \mu(\mu - 2\lambda)$. $\square$

We may think about $\lambda \neq 0$ as a "topological deviation" from the $\lambda = 0$ case. Therefore η is the most natural candidate for the principal curl eigenform, or the energy minimizer on Ξ_η. We note that Hopf fields are principal curl eigenfields and therefore energy minimizers. Lemma 4.2 and Proposition 5.2 show that in general we expect the first eigenvalue μ_1 to depend on the length of the fibre l, the spectrum on the surface and λ. Therefore η may not always be a principal curl eigenform.

Conjecture 5.3. *The curl eigenform η defined by a K-contact structure is always energy-minimizing on Ξ_η.*

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References

[1] V. Arnold and B. Khesin. *Topological Methods in Hydrodynamics*, volume 125 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[2] D. Bennequin. Entrelacements et équations de Pfaff. *Astérisque*, 87–161, Soc. Math. France, Paris, 1983.

[3] D. Blair. *Riemannian geometry of contact and symplectic manifolds*, volume 203 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2002.

[4] S. Chandrasekhar and P. Kendall. On force-free magnetic fields. *Astrophys. J.*, 126:457–460, 1957.

[5] C. Chicone. The topology of stationary curl parallel solutions of Euler's equations. *Israel J. Math.* 39:1-2, 161–166, 1981.

[6] S. S. Chern and R. S. Hamilton. On Riemannian metrics adapted to three-dimensional contact manifolds. In *Workshop Bonn 1984 (Bonn, 1984)*, volume 1111 of *Lecture Notes in Math.*, pages 279–308. Springer, Berlin, 1985.

[7] Y. Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.* 98:3, 623–637, 1989.

[8] J. Etnyre and R. Ghrist. Contact topology and hydrodynamics. I. Beltrami fields and the Seifert conjecture. *Nonlinearity*, 13(2):441–458, 2000.

[9] E. Giroux. Structures de contact sur les variétés fibrées en cercles audessus d'une surface. *Comment. Math. Helv.*, 76(2):218–262, 2001.

[10] R. Komendarczyk. On the contact geometry of nodal sets, `math.dg/0402070`. To appear, *Trans. of Amer. Math. Soc.*, 2004.

[11] L. Nicolaescu. Adiabatic limits of the Seiberg-Witten equations on Seifert manifolds. *Comm. Anal. Geom.*, 6(2):331–392, 1998.

[12] S. Rosenberg. *The Laplacian on a Riemannian manifold*, volume 31 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. An introduction to analysis on manifolds.

[13] K. Uhlenbeck. Generic properties of eigenfunctions. *Amer. J. Math.*, 98(4):1059–1078, 1976.

[14] Z. Yoshida and Y. Giga. Remarks on spectra of operator rot. *Math. Z.*, 204(2):235–245, 1990.

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