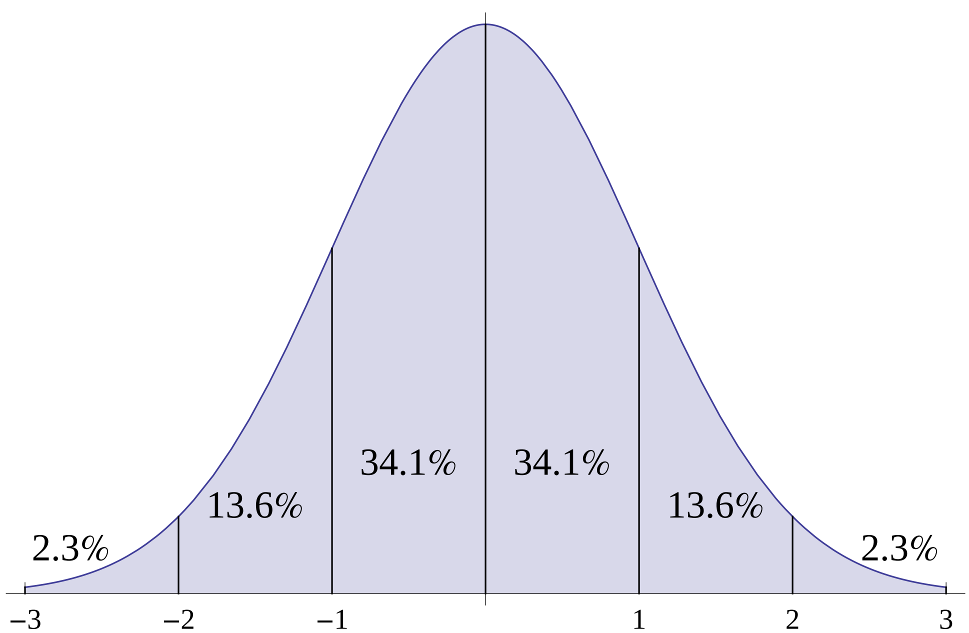
Instructions

(1) Write your name and sign in the space provided below.
(2) This is a closed-book exam.
(3) No calculators or cheat-sheets are allowed. Be sure to show all of your work in each problem.
(4) Whenever appropriate, *circle your answer*

Name: ______________________________



	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Possible	4	5	10	10	20	10	6	10	10	15	20	10	10	10
Score														

Total $\frac{\quad}{150}$

(4 pts) (1) In computer language, a *bit* is a value that can be either 0 or 1, and a *word* is an ordered set of 32 bits. How many possible words are there? *Hint:* draw a tree.

Solution: 2^{32}

(5 pts) (2) Given that there are 21 people in a classroom and 365 days in a year, what is the probability that two or more people in the classroom share the same birthday? Do not simplify your answer.

Solution: The probability that no two people share the same birthday is

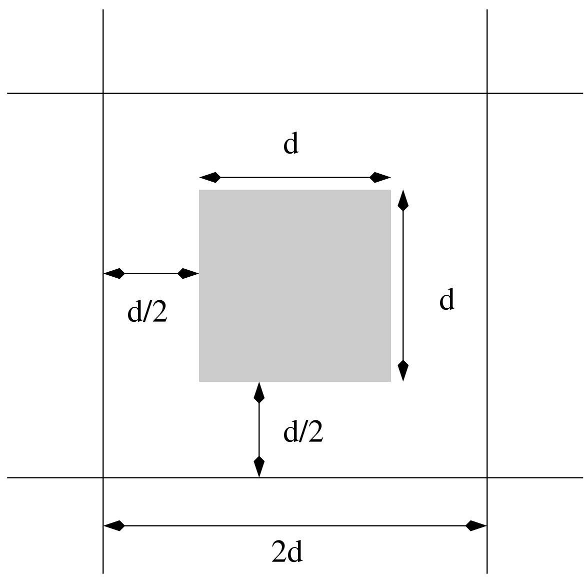
$$\frac{365 \times 364 \times \cdots \times 345}{365^{21}},$$

so the probability that two or more share a birthday is

$$1 - \frac{365 \times 364 \times \cdots \times 345}{365^{21}}.$$

(10 pts) (3) While visiting a carnival, you notice a coin-tossing game where quarters are tossed onto a checkerboard. The carnival keeps all quarters, but for each one landing entirely within one square of the checkerboard, the player wins a dollar. Assuming that the edge of each square is twice the diameter of a quarter and that the tosses are random, is the game fair? That is, is the player as likely to win money as to lose it?

Solution: First, compute the probability of winning: that is, the probability of the center of the coin landing in the unshaded region of a square of the checkerboard:



This probability of winning is

$$\frac{A_{\text{unshaded}}}{A_{\text{total}}} = \frac{2(d \times d/2) + 2d \times d/2}{(2d)^2} = \frac{d^2 + d^2}{4d^2} = 1/4,$$

so that the expected winnings is

$$E(X) = \frac{1}{4}(1\$ - 0.25\$) + \frac{3}{4}(-0.25\$) = 0$$

and the game is indeed fair.

(10 pts) (4) In a standard deck of cards (52 total cards, 13 numbered cards in each of four suits), what is the probability of getting a pair (different suits, same numerical value) in a poker hand of five randomly chosen cards? Put your answer in terms of binomial coefficients.

Solution:

$$P(\text{pair}) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}$$

Numerator explanation in order of terms:

(1) Choose the rank

(2) Choose the pair

(3) Choose the other ranks

(4) (last three) Choose the other cards (given ranks above)

(20 pts) (5) A fair six-sided die with faces labelled from one to six is rolled twice. What is the probability that the sum of the faces is *greater than seven*, given that

(a) the first outcome was a four?

(b) the first outcome was greater than three?

(c) the first outcome was a one?

(d) the first outcome was less than five?

Please show all of your work.

Solution:

(a)

$$\begin{aligned} P(X+Y>7|X=4) &= \frac{P(\{X+Y>7\}\cap\{X=4\})}{P(X=4,Y>3)} \\ &= \frac{3/36}{1/6} \\ &= \frac{1}{2}. \end{aligned}$$

(b)

$$\begin{aligned} P(X+Y>7|X>3) &= \frac{P(\{X+Y>7\}\cap\{X>3\})}{P(X>3)} \\ &= \frac{P\left(\left\{\begin{array}{c}(4,4),(4,5),(4,6),(5,3),(5,4),(5,5),\\(5,6),(6,2),(6,3),(6,4),(6,5),(6,6)\end{array}\right\}\right)}{3*6/36} \\ &= \frac{2}{3} \end{aligned}$$

(c)

$$P(X+Y>7|X=1)=0$$

(d)

$$\begin{aligned} P(X+Y>7|X<5) &= \frac{P(\{X+Y>7\}\cap\{X<5\})}{P(X<5)} \\ &= \frac{P\left(\{(4,4),(4,5),(4,6),(3,5),(3,6),(2,6)\}\right)}{2/3} \\ &= \frac{1}{4} \end{aligned}$$

(10 pts) (6) Let $X, Y : \Omega \to [0,1]$ be continuous random variables with uniform density functions. Set $Z = X + Y$ and compute the density function for Z.

Solution:

$$\begin{aligned} f_Z(z) &= f_X \star f_Y(z) \\ &= \int_0^1 f_X(x) f_Y(z-x)dx \\ &= \int_0^1 f_Y(z-x)dx \end{aligned}$$

If $0 \leq z \leq 1$, we have that

$$f_Z(z) = \int_0^z dx = z,$$

while if $1 \leq z \leq 2$, we have that

$$f_Z(z) = \int_{z-1}^1 dx = 2 - z.$$

Combining these results, we find that

$$f_Z(z) = \begin{cases} z, & \text{if } 0 \leq z \leq 1 \\ 2 - z, & \text{if } 1 \leq z \leq 2 \\ 0, & \text{else} \end{cases}$$

(6 pts) (7) For each chance experiment and random variable described below, choose the correct distribution function for the random variable and write the corresponding number in the space provided below. In both cases, $0 < p < 1$ and define $q = 1 - p$.

(a) Consider flipping a coin with probability p of flipping heads. Let X represent the number of tosses until the k^{th} head appears, and find $P(X = \ell)$ for $\ell \geq k$.

(b) Let X represent the number of heads in ℓ total tosses of an unfair coin with probability p of flipping heads. Find $P(X = k)$.

(1) $\binom{\ell-1}{k-1} p^{\ell} q^{\ell-k}$.

(2) $\binom{\ell}{k} p^k q^{\ell-k}$.

(a) _______________

(b) _______________

Solution:

(a) 1

(b) 2

(10 pts) (8) For each chance experiment and random variable described below, write down the probability $P(X = \ell)$.

(a) An urn contains $N - k$ distinguishable yellow balls and k distinguishable green balls. Each time a ball is withdrawn, its color is noted and it is set aside. If n balls are withdrawn in this fashion, let X represent the number of green balls and find $P(X = \ell)$ – you need not simplify your answer.

(b) Consider an infinite Bernoulli trials process with probability p of success on each trial. Let X be the number of trials up to and including the first success, and find $P(X = \ell)$.

Solution:

(a) A hypergeometric distribution with

$$P(X = \ell) = \frac{\binom{k}{\ell}\binom{N-k}{n-\ell}}{\binom{N}{n}}.$$

(b) A geometric distribution; for $q = 1 - p$,

$$P(X = \ell) = q^{\ell-1}p.$$

(10 pts) (9) Exactly one of six similar keys opens a certain door. If you try the keys, one after another, what is the expected number of keys that you will have to try before success? Show your work.

Solution: 7/2

(15 pts) (10) Let $X : \Omega \to [-1, 1]$ be a continuous random variable with density function $f(x)$. Find the expected value and variance of X if for $|x| < 1$,

(a) $f(x) = 1/2$

(b) $f(x) = |x|$

(c) $f(x) = (3/8)(x + 1)^2$

Solution:

(a)

$$
\begin{aligned}
E(X) &= \int_{-1}^{1} x f(x) \\
&= \int_{-1}^{1} x/2 \\
&= \frac{x^2}{4}\bigg|_{-1}^{1} \\
&= 0 \\
V(X) &= \int_{-1}^{1} x^2 f(x) - E(X)^2 \\
&= \int_{-1}^{1} x^2/2 \\
&= \frac{x^3}{6}\bigg|_{-1}^{1} \\
&= 1/3
\end{aligned}
$$

(b) $E(X) = 0$, $V(X) = 1/2$

(c) $E(X) = 1/2$, $V(X) = 3/20$

(20 pts) (11) Let $\{X_k\}_{k\in\mathbb{N}}$ be an independent trials process, with $E(X_i) = 30$ and $\sigma(X_i) = 5$ for all $i \in \mathbb{N}$. Define $S_n = \sum_{i=1}^{n} X_i$ and answer the following:

(a) Compute the expected value of S_n

(b) Compute the standard deviation of S_n

(c) Use the central limit theorem to estimate $P(2950 \leq S_{100} \leq 3050)$

(d) Use the central limit theorem to estimate $P(2900 \leq S_{100} \leq 3000)$

Solution:

(a) $E(S_n) = \sum_{i=1}^{n} E(X_i) = 30n$

(b) $\sigma(S_n) = \sqrt{\sum_{i=1}^{n} V(X_i)} = \sigma(X_i)\sqrt{n} = 5\sqrt{n}$

(c) This corresponds to $P(-1 \leq S_{100}^* \leq 1) \approx 68.2\%$.

(d) This corresponds to $P(-2 \leq S_{100}^* \leq 0) \approx 47.7\%$.

(10 pts) (12) A one-dollar bet on craps has an expected winning of -0.0141 with variance 0.05. If a gambler makes a large number of sequential of one-dollar bets, which of the following *best* describes what the law of large numbers says about his bets? *Briefly* justify your answer.

(a) In the long run, the probability that he loses money tends to zero and his expected winnings are small.

(b) In the long run, the probability that he loses tends to zero and his expected winnings are large.

(c) In the long run, the probability that he wins money tends to zero and his expected losses are small.

(d) In the long run, the probability that he wins money tends to zero and his expected losses are large.

(e) If A_n is his average winnings after n bets, then

$$P\big(|A_n + 0.0141| \geq 1\big) \leq 0.05.$$

Solution: The answer is (d).

The law of large numbers says that for all $\epsilon > 0$,

$$\lim_{n\to\infty} P\big(|S_n/n + 0.0141| > \epsilon\big) = 0.$$

This means that the probability that his winnings are positive tends to zero. Say X_i is the random variable measuring the amount won on the i^{th} roll. Then $E(X_i) = -0.0141$, and the expected winnings after n rolls is given by the expected value of $S_n = \sum_{i=1}^{n} X_i$: $E(S_n) = nE(X_i) = -0.0141n$. This number grows with n, so for a large number of bets his expected losses are large.

Part (e) is the Chebyshev inequality, not the law of large numbers.

(10 pts) (13) Consider a discrete random variable X. For each distribution function below, compute the moment generating function $g(t)$ and use *it* to find the expected value of X.

The top row of the matrices denotes the possible values of X, and the bottom row the corresponding probability (i.e. for part (a), $P(X = 2) = 1/4$).

(a) $P = \begin{pmatrix} 0 & 1 & 2 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$

(b) $P = \begin{pmatrix} -3 & -2 & -1 & 1 & 2 \\ 1/8 & 1/8 & 0 & 3/8 & 3/8 \end{pmatrix}$

Solution:

(a)

$$\begin{aligned} g(t) &= E(e^{tX}) \\ &= \frac{1}{4}e^0 + \frac{1}{2}e^t + \frac{1}{4}e^{2t} \\ &= \frac{1 + 2e^t + e^{2t}}{4} \\ E(X) &= g'(0) \\ &= \frac{1}{4}\left(2e^t + 2e^{2t}\right)|_{t=0} \\ &= 1 \end{aligned}$$

(b)

$$\begin{aligned} g(t) &= \frac{1}{8}\left(e^{-3t} + e^{-2t} + 3e^t + 3e^{2t}\right) \\ E(X) &= \frac{1}{8}(-3 - 2 + 3 + 6) \\ &= \frac{1}{2} \end{aligned}$$

(10 pts) (14) Consider a drunken student walking up a hill with 6 blocks. The student lives at the top of the hill on 6th street and drinks at a bar at the bottom of the hill on 1st street.

At each intervening block (2nd, 3rd, 4th, and 5th), the student walks uphill one block with a probability p or downhill one block with probability q, for $0 < p < 1$ and $q = 1 - p$. If the student reaches home or the bar, he stays there.

Construct a Markov Chain (S, P) describing this situation, with $S = (6, 5, 4, 3, 2, 1)$. Is this an absorbing Markov chain? Justify your answer.

Solution:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ p & 0 & q & 0 & 0 & 0 \\ 0 & p & 0 & q & 0 & 0 \\ 0 & 0 & p & 0 & q & 0 \\ 0 & 0 & 0 & p & 0 & q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It is an absorbing Markov chain because it is always possible to reach the bar or home (the absorbing states) from any intervening block.