A SHORT INTRODUCTION TO SIMPLE LIE ALGEBRA REPRESENTATIONS

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Abstract. This is a (very) short introduction to simple Lie algebra representations, based on Di Francesco et al.[FMS97], chapter 13. These notes are meant to span two one-hour lectures, perhaps expanded into three with examples. See references for more mathematical presentations.

Contents

1. Introduction

Whenever the phrase "continuous symmetry" appears in physics, the concept of a Lie algebra is behind it. Especially ubiquitous are the simple Lie algebras, associated with groups like $U(N)$, $SU(N)$, $SO(N)$, $Sp(N)$, etc, as well as semi-simple Lie algebras, which are composed of simple Lie algebras.

More important than the algebras themselves are their representations. These show up as descriptions of a set of operators acting on a Hilbert space, for instance. We will describe a simple Lie algebra representation completely by its weights, as well as the roots of the Lie algebra itself. The weights of the representation are encoded in Dynkin diagrams.

For a more mathematical approach to simple Lie algebras, including proofs of existence and uniqueness of the Cartan subalgebra, proofs of the number of simple roots, and more results on roots and weights, see [FH91] and [Kna02][1].

2. Simple Lie algebras

We begin with a series of definitions;

Definition 1. *Lie algebra* - A vector space X together with an antisymmetric bilinear product on X, usually written as $[\ ,\ ] : X \times X \to X$, which satisfies the Jacobi Identity;

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

Definition 2. *Lie algebra representation* - A homomorphism ϕ from a Lie algebra $\mathfrak{g}$ to $\mathrm{End}(V)$, the space of linear transformations of a vector space V. $\phi : \mathfrak{g} \to \mathrm{End}(V)$ respects the bilinear product.

Definition 3. *Simple* - A Lie algebra is simple if it is non-abelian and has no non-zero proper ideals.

[1] Commutative diagrams in this paper were typeset using Paul Taylor's diagrams package - `http://www.cs.man.ac.uk/~pt/diagrams/`

Definition 4. *Semi-Simple* - A Lie algebra is semi-simple if it is non-abelian and has no non-zero proper abelian ideals. This implies that the algebra may be written as $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$, where $\mathfrak{g}_i$ are all simple.

Definition 5. *Dimension* - A representation of a Lie algebra is said to have dimension n if the vector space V has $\dim_{\mathbb{K}} V = n$, where $\mathbb{K}$ is the field over which the vector space is defined (usually taken to be $\mathbb{R}$ or $\mathbb{C}$).

Definition 6. *Irreducible* A representation is said to be irreducible if $\{0\}$ and V are the only invariant subspaces. That is, if $W \subseteq V$ is a subspace, $h \in \mathfrak{g}$, and $\phi(h) \cdot W \subseteq W$, then $W = \{0\}$ or $W = V$. If a representation has finite dimension, irreducible and simple are equivalent.

Definition 7. *Generators* A set of generators for a Lie algebra is a basis for the vector space underlying the algebra (#generators $= \dim_{\mathbb{K}} \mathfrak{g}$).

For simple Lie algebras, one can always choose a particularly convenient basis called the Cartan-Weyl basis. In this basis, there exists a maximal subset of *abelian generators*, i.e. ones that satisfy

$$[H^i, H^j] = 0 \qquad \text{for all } i, j.$$

This set of generators forms a basis for the *Cartan Subalgebra* $\mathfrak{h} \subseteq \mathfrak{g}$. The rest of the basis vectors may be arranged in linear combinations E^α, which satisfy

$$[H^i, E^\alpha] = \alpha^i E^\alpha$$

for some number α^i. If we write

$$\begin{pmatrix} [H^1, E^\alpha] \\ [H^2, E^\alpha] \\ \vdots \\ [H^r, E^\alpha] \end{pmatrix} = \begin{pmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^r \end{pmatrix} E^\alpha = \alpha E^\alpha$$

α is called a *root*, and E^α is called a *ladder operator*. The Lie algebra then decomposes as

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where Δ is the set of roots and $\mathfrak{g}_\alpha$ is the subspace generated by E^α.

Definition 8. *Rank* - The rank of a Lie algebra $\mathfrak{g}$ is the dimension of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

The Cartan subalgebra $\mathfrak{h}$ is the maximal subalgebra of $\mathfrak{g}$; there does not exist a subalgebra $\mathfrak{f} \subset \mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{f}$. This implies that the roots are non-degenerate. Furthermore, note that there is a natural pairing $< \alpha, H^i >= \alpha^i$. Thus, we see that α is an element of the dual of the Cartan subalgebra, $\alpha \in \mathfrak{h}^\vee$. In addition, notice that if α is a root, then $-\alpha$ is as well:

$$\begin{aligned} \Big([H^i, E^\alpha] = \alpha^i E^\alpha\Big)^\dagger \\ [(E^\alpha)^\dagger, H^i] = (\alpha^i)^\dagger (E^\alpha)^\dagger \\ [H^i, (E^\alpha)^\dagger] = -\alpha^i (E^\alpha)^\dagger, \end{aligned}$$

since H^i hermitian implies that their eigenvalues are real. Thus, we have that $E^{-\alpha} = (E^\alpha)^\dagger$, and we see that E^α cannot be hermitian.

The roots can also be thought of as the eigenvalues of the H^i in the *adjoint representation*;

$$\begin{aligned} \phi(H^i) E^\alpha &= \mathrm{ad}_{H^i} E^\alpha \\ &\equiv [H^i, E^\alpha] \\ &= \alpha^i E^\alpha \end{aligned}$$

So, writing $E^\alpha = |\alpha\rangle$, we have that $H^i |\alpha\rangle = \alpha^i |\alpha\rangle$. Note further that

$$\begin{aligned} [H^i, [E^\alpha, E^\beta]] &= [[H^i, E^\alpha], E^\beta] + [[E^\beta, H^i], E^\alpha] \\ &= \alpha^i [E^\alpha, E^\beta] + \beta^i [E^\alpha, E^\beta] \\ &= (\alpha^i + \beta^i)[E^\alpha, E^\beta]. \end{aligned}$$

Thus;

$$\begin{aligned} &\text{if } \alpha + \beta \text{ is a root, } [E^\alpha, E^\beta] \sim E^{\alpha+\beta} \\ &\text{if } \alpha = -\beta, \text{ then } [E^\alpha, E^\beta] \sim H^j \text{ for some j} \end{aligned}$$

3. Killing Form

There exists a symmetric bilinear form on any Lie algebra, which is non-degenerate for simple Lie algebras. It is called the *Killing Form*, and is defined by

$$K(X,Y) = \frac{1}{2g}\mathrm{Tr}[\mathrm{ad}X\mathrm{ad}Y],$$

where g is a constant to be defined later. $\mathrm{ad}X\mathrm{ad}Y$ is an operator on the Lie algebra: for example, take the algebra defined by two generators X and Y, with $[X,Y]=Y$. Then the matrix presentation of $\mathrm{ad}X\mathrm{ad}X$ is

$$\left.\begin{array}{l}\mathrm{ad}X\mathrm{ad}X(X) = [X,[X,X]] = 0\\ \mathrm{ad}X\mathrm{ad}X(Y) = [X,[X,Y]] = Y\end{array}\right\} \Rightarrow K(X,X) = \frac{1}{2g}\mathrm{Tr}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{2g}$$

Indeed, the Cartan subalgebra is orthogonal with respect to the Killing form; $K(H^i,H^j)=\delta^{ij}$. Since it is non-degenerate for simple Lie algebras, the Killing form may be used to define the dual space of the Cartan subalgebra. That is,

$$K(—,H^\gamma) \equiv \gamma \in \mathfrak{h}^\vee.$$

γ's action on an element of the Cartan subalgebra is then defined as $\gamma(H^i) \equiv K(H^i,H^\gamma)$, and it forms an inner product on the dual space $\mathfrak{h}^\vee$ by $(\gamma,\beta) \equiv K(H^\gamma,H^\beta)$. Here, H^γ denotes the sum $\gamma\cdot H = \sum_i \gamma^i H^i$. With this information, and defining $|\alpha|^2 = (\alpha,\alpha)$, we normalize the ladder operators as follows

$$[E^\alpha, E^\beta] = N_{\alpha\beta}E^{\alpha+\beta}$$

$$[E^\alpha, E^{-\alpha}] = \frac{2}{|\alpha|^2}\alpha\cdot H,$$

where $N_{\alpha\beta}$ are numbers associated with the specific algebra. We can also define the *coroot* associated with a root α;

$$\alpha_i^\vee = \frac{2\alpha_i}{|\alpha_i|^2}.$$

4. Weights

So far, we have been dealing with a specific representation, the adjoint representation of $\mathfrak{g}$ onto itself. For a general, finite dimensional representation, we can find a basis for the representation vector space V such that

$$H^i\left|\lambda\right\rangle = \lambda^i\left|\lambda\right\rangle.$$

The collection $\{\lambda^i\} = \lambda$ is called a *weight*. Clearly, these weights are in the dual space of the Cartan subalgebra. They are called roots in the adjoint representation.

Now, we come to the reason E^α are called raising operators.

$$\begin{aligned} H^iE^\alpha\left|\lambda\right\rangle &= ([H^i,E^\alpha] + E^\alpha H^i)\left|\lambda\right\rangle \\ &= (\lambda^i+\alpha^i)\left|\lambda\right\rangle \\ \Rightarrow E^\alpha &\sim \left|\lambda+\alpha\right\rangle \end{aligned}$$

Now, since we are dealing with a finite dimensional representation, repeated application of E^α must be a nilpotent operation. That is, there exist integers p and q such that

$$(E^\alpha)^{p+1}\left|\lambda\right\rangle \sim E^\alpha\left|\lambda+p\alpha\right\rangle = 0$$

$$(E^{-\alpha})^{q+1}\left|\lambda\right\rangle \sim E^{-\alpha}\left|\lambda-q\alpha\right\rangle = 0,$$

for any root α. Also, note that for any root α, there exists an su(2) subalgebra generated by $\{E^\alpha, E^{-\alpha}, 1/|\alpha|^2\alpha\cdot H\}$. By manipulating this subalgebra, and using the fact that we are dealing with a finite dimensional representation, it is easy to show that for any weight λ and any root α, $\frac{2}{|\alpha|^2}(\alpha,\lambda)\in\mathbb{Z}$.

5. Simple Roots

Once we know a full set of generators of the Lie algebra, we can express any element, and even the Lie bracket, in terms of that basis. For simple Lie algebras, there is an especially nice basis composed of the generators of the Cartan subalgebra H^i, and the set of simple roots.

Let r be the rank of $\mathfrak{g}$, and let $(\beta_1,\beta_2,\ldots,\beta_r)$ be an ordered basis for $\mathfrak{h}^\vee$. Any root α may be expanded as $\alpha=\sum_i n_i\beta_i$.

Definition 9. *Positive root* - α is called a positive root if the first nonzero number in the sequence $n_1,\ldots,n_r$ is greater than zero.

Denote the set of positive roots by Δ_+ and the set of negative roots by Δ_-. Note further that since $-\alpha$ is a root if α is, $\Delta_+ = -\Delta_-$.

Definition 10. *Simple root* - A root is simple if it cannot be written as the sum of two positive roots.

It turns out that there are exactly r simple roots. Also, note that if α_i and α_j are simple roots, then their difference $\alpha_i - \alpha_j$ is not. If it were, then we could write $\alpha_i = \alpha_j + (\alpha_i - \alpha_j)$, which is a contradiction. (Here the subscripts do not refer to vector components, but to a vector in a list of r vectors.)

There is a distinguished root, the highest root θ. Like any other root, it can be expanded in the basis of simple roots. Its coefficients in this basis have the special name *marks*, a_i. In the dual basis, the coefficients are called *comarks*, $a_i^\vee$;

$$\theta = \sum_{i=1}^{r} a_i \alpha_i = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee.$$

These comarks are used to define the constant g mentioned earlier: it is known as the *dual Coxeter number*;

$$g = \sum_i a_i^\vee + 1$$

This number arises in the regularization of Chern Simons theories.

6. Cartan Matrix

A very important description of simple Lie algebras comes from the Cartan matrix. Its elements are defined as

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{|\alpha_j|^2}.$$

Note that it is in general not symmetric, and that the diagonal entries are all $A_{ii} = \frac{2(\alpha_i, \alpha_i)}{|\alpha_i|^2} = 2$. As we saw earlier, these elements are in fact all integers! Let us list some nice properties of the Cartan matrix elements.

$$\begin{aligned} A_{ij} &\in \mathbb{Z} \\ A_{ij} &\leq 0 && (i \neq j) \\ A_{ij}A_{ji} &= \{0, 1, 2, 3\} && (i \neq j) \\ A_{ij} = 0 &\Rightarrow A_{ji} = 0 \\ \det A &\neq 0 \end{aligned}$$

It is clear that both roots and weights are in $\mathfrak{h}^\vee$. However, the number of weights is the dimension of the representation minus its rank, so the weights may be expanded in terms of the simple roots. To this end, a basis is chosen which is dual to the coroots.

$$(\alpha_i^\vee, \omega_j) = \delta_{ij}.$$

The set of vectors $\omega_1, \ldots, \omega_r$ are called the fundamental weights. As the fundamental weights form a basis, any weight may be expanded in terms of the fundamental weights.

Definition 11. *Dynkin labels* - the coefficients $\{\lambda_i\}$ of the expansion of a weight λ in the basis of fundamental weights are called Dynkin labels.

$$\lambda = \sum_i \lambda_i \omega_i$$

For finite dimensional irreducible representations, the Dynkin labels λ_i are always integers.

There is also a special weight called the Weyl vector, whose Dynkin labels are all 1; $\rho = \sum_i \omega_i$. It turns out that this vector can also be written in terms of the positive roots;

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.$$

7. Dynkin Diagrams

Consider a set of simple roots $\{\alpha_1, \cdots, \alpha_r\}$. For each simple root, draw a vertex and associate to it a *weight* proportional to $|\alpha_i|^2$. Connect the verticies as follows;

(1) Choose two verticies, α_i, α_j
(2) Draw $A_{ij}A_{ji}$ edges (lines) between them

If the graph produced by these steps is connected, the associated algebra is irreducible. Generally Dynkin diagrams are ordered left to right by increasing weight.

We can also recover the Cartan matrix from a Dynkin diagram. Call the weights $\{\omega_i\}$.

(1) if there are no edges from the i^{th} vertex to the j^{th} vertex, then $A_{ij}A_{ji} = 0$, and thus $A_{ij} = A_{ji} = 0$ by the properties of Cartan matrix elements.

(2) Suppose there exist edges between i and j. It turns out that

$$\frac{A_{ij}}{A_{ji}} = \frac{\omega_j}{\omega_i},$$

and so together with the fact that $A_{ij}A_{ji}$ is the number of edges, A_{ij} and A_{ji} are determined completely.

8. The Weyl Group

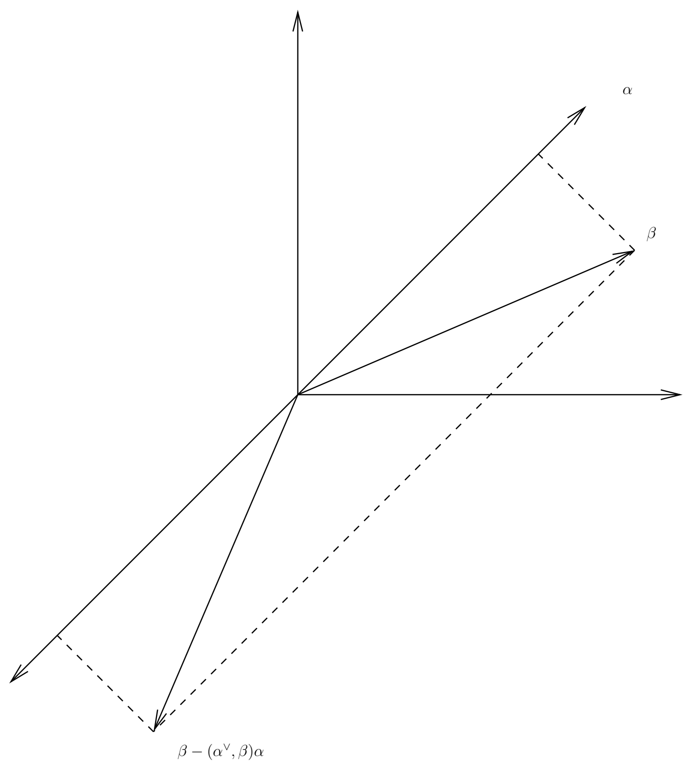
Consider a state $|\beta\rangle \equiv E^{\beta}$. Define the eigenvalue of the operator $\frac{1}{|\alpha|^2}\alpha \cdot H$ corresponding to the eigenstate $|\beta\rangle$ to be m. That is,

$$\begin{aligned} \frac{1}{|\alpha|^2}\alpha \cdot H\,|\beta\rangle &= m\,|\beta\rangle \\ &\equiv \frac{\alpha_i}{|\alpha_i|^2}\mathrm{ad}(H^i)E^{\beta} \\ &= \frac{\alpha_i}{|\alpha_i|^2}[H^i, E^{\beta}] \\ &= \frac{\alpha_i}{|\alpha_i|^2}\beta^i E^{\beta} \\ &= \frac{1}{2}(\alpha^{\vee}, \beta^i)\,|\beta\rangle \end{aligned}$$

So, we see that $2m = (\alpha^{\vee}, \beta)$. Now, for $m \neq 0$, there exists another state with eigenvalue $-m$, corresponding to $E^{-\beta}$. It is easy to show that this state is in fact defined by the equation

$$\beta - (\alpha^{\vee}, \beta)\alpha.$$

This equation simply defines a reflection in the vector space $\mathfrak{h}^{\vee}$ about the hyperplane perpendicular to α.



For each simple root α_i, we define a reflection s_i, whose action on a state β is given by

$$s_i\beta = \beta - (\alpha_i^{\vee}, \beta)\alpha_i.$$

These reflections are called *simple Weyl reflections*, and together they generate a group on r letters, the *Weyl group* $\mathcal{W}$ of the algebra. A group on r letters means that any element $w \in \mathcal{W}$ can be written as

$$w = s_i s_j \cdots s_k.$$

This group would be freely generated, except that the Weyl reflections (letters) obey some relations;

$$s_i^2 = 1 \qquad \text{and} \qquad s_i s_j = s_j s_i \text{ if } A_{ij} = 0,$$

which can be generalized to

$$(s_is_j)^{m_{ij}} = 1 \qquad \text{where } m_{ij} = \begin{cases} 1 & i = j \\ \frac{\pi}{\pi - \theta_{ij}} & i \neq j \end{cases}. \tag{1}$$

Here θ_{ij} is the angle between the roots α_i and α_j. The action of the Weyl letters on the simple roots is $s_i\alpha_j = \alpha_j - A_{ji}\alpha_i$, which extends to an action on the weights; $s_i\lambda = \lambda - (\alpha^\vee, \lambda)\alpha$. One can also easily show that inner product on the weights is invariant under the action of the Weyl group. That is

$$(s_i\lambda, s_i\beta) = (\lambda, \beta).$$

We can use the Weyl group to break up the root space into *chambers*. The number of chambers is equal to the order (number of elements) of the group.

Definition 12. *Weyl Chamber* - The chamber of an element w of the Weyl group is defined to be the set of vectors λ in the weight space such that $(w\lambda, \alpha_i) \geq 0$ for all i:

$$C_w = \{\lambda \in \mathfrak{h}^\vee | (w\lambda, \alpha_i) \geq 0 \,\forall\, i\}$$

The Weyl chamber of the identity element in $\mathcal{W}$ is called the *fundamental chamber*. Consider the root system and Weyl chambers for the algebra $\mathfrak{su}(3)$.

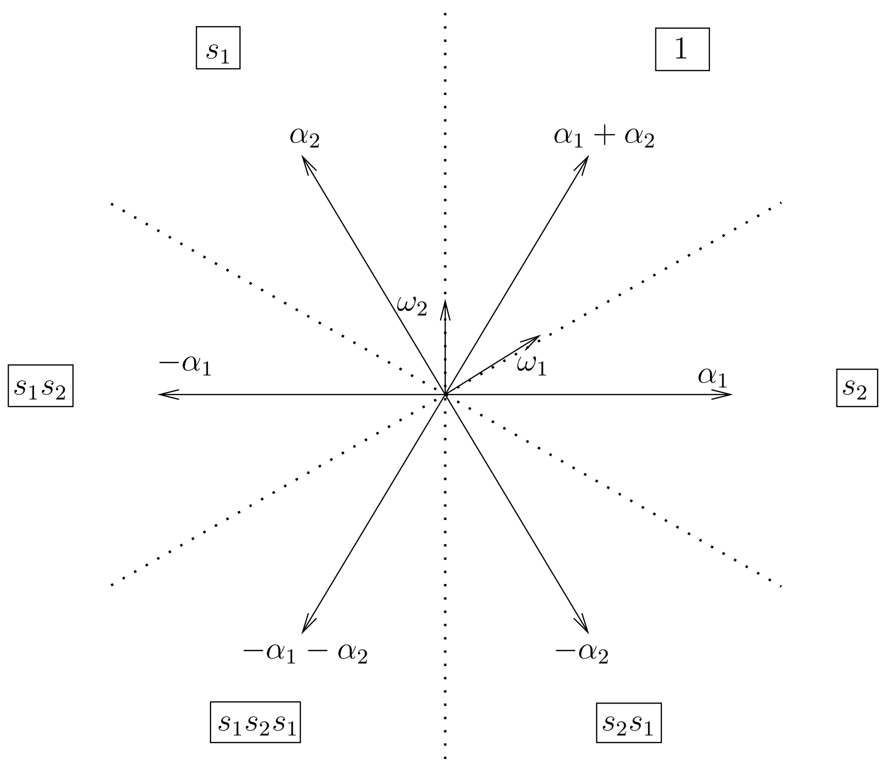


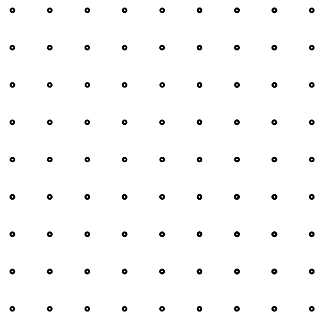
Figure 1. α_i are simple roots, ω_i are fundamental weights, and the boxed letters are elements of the Weyl group. Adapted from [FMS97].

9. Lattices

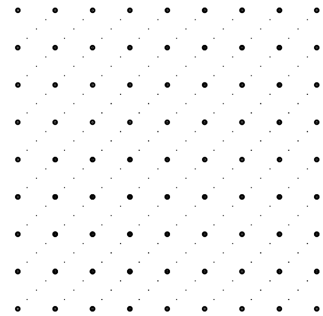
A lattice is a set of points spaced by integers. A lattice is isomorphic to $\mathbb{Z}^n$, but not canonically. For example, Lattice 1 is generated by the vectors $(1, 0), (0, 1)$, while lattice 2 is generated by the vectors $(0, 1), (\frac{1}{4}, \frac{1}{4})$, though both are isomorphic to $\mathbb{Z}^2$. For simple Lie algebras, there are three important r-lattices.

the weight lattice	$P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_r$
the root lattice	$Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_r$
the coroot lattice	$Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_r^\vee$

Since the weights of finite dimensional representations have integer Dynkin labels, they are in P. It turns out that the integers which specify the position of a weight in P are the eigenvalues of the Cartan subalgebra generators H^i in a special basis called the Chevalley basis.



Lattice 1



Lattice 2

10. Highest Weight Representations

Any finite-dimensional irreducible representation has a unique highest weight state. For definiteness, refer to an arbitrary representation $\phi : \mathfrak{g} \to \mathrm{End}(V)$ Consider the highest weight state $|\lambda\rangle \in V$, whose defining property is

$$E^{\alpha} |\lambda\rangle = 0 \; \forall \; \alpha > 0 \;\; (\alpha \in \Delta_+). \tag{2}$$

Note that this implies that the Dynkin labels are all positive. Listed below are some nice properties of highest weights.

(i) The highest weight state $|\lambda\rangle$ is independent of the ordering chosen to define the simple roots.
(ii) The weight space which corresponds to λ, $V_\lambda \subset V$ is one dimensional (i.e. it is uniquely determined by λ)
(iii) The vectors in V_λ are the only ones which satisfy (2)

Weights whose Dynkin labels are all positive are called dominant. Each dominant weight λ corresponds to a unique finite dimensional irreducible representation, denoted L_λ. Each weight in the representation is generated by the action of all possible combinations of negative roots on $|\lambda\rangle$;

$$\mathsf{L}_\lambda = \left\{ E^{\alpha} E^{\beta} \cdots E^{\zeta} |\lambda\rangle \,|\, \{\alpha, \beta \cdots, \zeta\} \in \Delta_- \right\} \tag{3}$$

The set of weights in L_λ is called the weight system Ω_λ. Since weights $\lambda' \in \Omega_\lambda$ were obtained by the action of the lowering operators, we will have the easy consequence

$$\lambda' \in \Omega_\lambda \;\Rightarrow\; \lambda - \lambda' \in \Delta_+.$$

In fact, we can write $\lambda' = \lambda - \sum_i n_i \alpha_i$, where the n_i are integers and α_i are simple roots. These facts will provide a systematic way to find all the weights in a given irreducible representation; begin with the highest weight $\lambda = \{\lambda_i\}$, and for each λ_i construct a sequence of weights

$$\lambda - \alpha_i, \lambda - 2\alpha_i, \cdots, \lambda - \lambda_i \alpha_i.$$

This method yields all weights in a representation L_λ, since we can write $[E^{\alpha}, E^{\beta}] \sim E^{\alpha+\beta}$, and the product of two negative roots is again negative.

Indeed, there is a lowest weight state, $\omega_\circ \lambda$, where $\omega_\circ$ is the longest element of the Weyl group. The negative of this state is the highest weight of the *conjugate representation*: $\lambda^* = -\omega_\circ \lambda$. It is easy to see that $\Omega_{\lambda^*} = -\Omega_\lambda$.

11. Universal Enveloping Algebra

Before discussing Quadratic Casimirs of Lie algebras, it behooves us to make a brief aside about the Universal Enveloping Algebra $U(\mathfrak{g})$ corresponding to a Lie algebra $\mathfrak{g}$. We motivate its construction by considering the composition of linear transformations of a vector space. If M, N, O, P are matrices acting on some vector space, with $|\lambda\rangle$ a vector, the expression $MNOP|\lambda\rangle$ is well defined.

Now, consider the case when these matrices are part of a representation of a Lie algebra: $M = \phi(\mathfrak{m}), N = \phi(\mathfrak{n}), O = \phi(\mathfrak{o}), P = \phi(\mathfrak{p})$, where $\mathfrak{m}, \mathfrak{n}, \mathfrak{o}, \mathfrak{p} \in \mathfrak{g}$. It would be sensible to write something of the form

$$\begin{aligned} MNOP|\lambda\rangle &= \phi(\mathfrak{m})\phi(\mathfrak{n})\phi(\mathfrak{o})\phi(\mathfrak{p})|\lambda\rangle \\ &= \widetilde{\phi}(\mathfrak{m} \otimes \mathfrak{n} \otimes \mathfrak{o} \otimes \mathfrak{p})|\lambda\rangle \end{aligned}$$

That is, we would like to define a map from the tensor algebra

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} T^n(\mathfrak{g})$$

to $\mathrm{End}(V)$. The problem is that $T(\mathfrak{g})$ doesn't know about the structure of the Lie algebra; it does not necessarily respect the Lie bracket.

$$\mathfrak{m} \otimes \mathfrak{n} \otimes \mathfrak{o} \otimes \mathfrak{p} - \mathfrak{n} \otimes \mathfrak{m} \otimes \mathfrak{o} \otimes \mathfrak{p} \neq [\mathfrak{m}, \mathfrak{n}] \otimes \mathfrak{o} \otimes \mathfrak{p}.$$

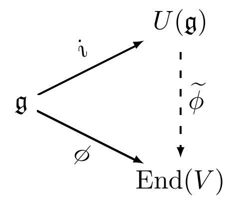
Thus, we should impose the commutation relations of the algebra on $T(\mathfrak{g})$; we write $T(\mathfrak{g})/J$, where J is the two-sided ideal generated by elements of the form $X \otimes Y - Y \otimes X - [X, Y]$, for $X, Y \in \mathfrak{g}$.

Definition 13. *Universal Enveloping Algebra* - We define the Universal Enveloping Algebra to be $U(\mathfrak{g}) = T(\mathfrak{g})/J$.

All this means is that if we have an expression in $T(\mathfrak{g})$ of the form $\mathfrak{m} \otimes \mathfrak{n} \otimes \mathfrak{o} \otimes \mathfrak{p} - \mathfrak{n} \otimes \mathfrak{m} \otimes \mathfrak{o} \otimes \mathfrak{p}$, it is in the same equivalence class as $[\mathfrak{m}, \mathfrak{n}] \otimes \mathfrak{o} \otimes \mathfrak{p}$ in the enveloping algebra.

$$\widetilde{\phi}\left(\mathfrak{m} \otimes \mathfrak{n} \otimes \mathfrak{o} \otimes \mathfrak{p} - \mathfrak{n} \otimes \mathfrak{m} \otimes \mathfrak{o} \otimes \mathfrak{p}\right) = \widetilde{\phi}\left([\mathfrak{m}, \mathfrak{n}] \otimes \mathfrak{o} \otimes \mathfrak{p}\right) = [[\mathfrak{m}, \mathfrak{n}] \otimes \mathfrak{o} \otimes \mathfrak{p}] \in U(\mathfrak{g})$$

Although it doesn't immediately follow from the definition, it can be shown that if $\phi : \mathfrak{g} \to \mathrm{End}(V)$ is a finite-dimensional irreducible representation, there exists a unique extension $\widetilde{\phi}$ such that the following diagram commutes:



12. Quadratic Casimir

We can define an element of $U(\mathfrak{g})$ called a quadratic casimir;

$$Q = \sum_{a,b} K(L^a, L^b)^{-1} L^a L^b,$$

where $\{L^a\}$ is a basis for the algebra. Note that $[Q, L^a] = 0$ for all a. The quadratic casimir takes nicer form in certain bases; for example, in the orthonormal basis $\{J^a\}$ $Q = \sum_a J^a J^a$. In the Cartan-Weyl basis, we will have

$$Q = \sum_i H^i H^i + \sum_{\alpha \in \Delta_+} \frac{|\alpha|^2}{2} (E^\alpha E^{-\alpha} + E^{-\alpha} E^\alpha).$$

Since $[Q, E^\alpha] = 0$ for all α, the eigenvalue of Q is the same for all states in an irreducible representation. Let us find the eigenvalue by expressing Q in the Cartan-Weyl basis, and acting on a highest weight state λ;

$$\begin{aligned} Q\left|\lambda\right\rangle &= \left(\sum_i H^i H^i + \sum_{\alpha \in \Delta_+} \frac{|\alpha|^2}{2} (E^\alpha E^{-\alpha} + E^{-\alpha} E^\alpha)\right)\left|\lambda\right\rangle \\ &= \left(\sum_i H^i H^i + \sum_{\alpha \in \Delta_+} H^\alpha + \frac{|\alpha|^2}{2} \sum_{\alpha \in \Delta_+} E^{-\alpha} E^\alpha)\right)\left|\lambda\right\rangle \\ &= \left(\sum_i H^i H^i + 2H^\rho\right)\left|\lambda\right\rangle \\ &= \left[|\lambda|^2 + 2(\lambda, \rho)\right]\left|\lambda\right\rangle. \end{aligned} \tag{4}$$

Here ρ is the Weyl vector. Since both λ and ρ are dominant, we see that $|\lambda|^2 + 2(\lambda, \rho) \geq 0$. In the adjoint representation, it turns out that $|\lambda|^2 + 2(\lambda, \rho) = 2g$.

13. Representation Index

For orthonormal generators, we have the relation $\mathrm{Tr}(J^a, J^b) = \delta^{a,b}$. How can we express this relationship in some representation λ? The trace of the generators will be proportional to the Kronecker delta, with constant of proportionality given by the *index of the representation* x_λ.

$$\mathrm{Tr}_\lambda\left(\phi_\lambda(J^a), \phi_\lambda(J^b)\right) = 2x_\lambda \delta^{a,b}$$

The index may also be expressed as

$$x_\lambda = \mathrm{Tr}_\lambda(Q) = \frac{\dim \mathsf{L}_\lambda[|\lambda|^2 + 2(\lambda, \rho)]}{2 \dim \mathfrak{g}}$$

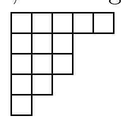
In the adjoint, we will have $x_\lambda = g$.

14. Young Tableaux

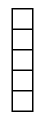
A *Young Tableau* is a diagrammatic representation of the highest weights of a Lie algebra. Let us concentrate on su(N). We can specify a representation by Dynkin labels $\lambda = \{\lambda_i\}$, or by a *partition*;

$$\lambda = \begin{pmatrix} \lambda_1 + \lambda_2 + \cdots + \lambda_N \\ \lambda_2 + \lambda_3 + \cdots + \lambda_N \\ \vdots \\ \lambda_N \end{pmatrix} \qquad \begin{array}{c} \lambda = \{\ell_1; \ell_2; \ldots; \ell_N\} \\ \ell_i = \sum_{j=i}^{N} \lambda_i \end{array}$$

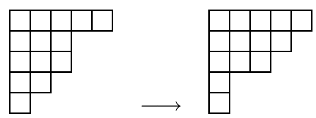
Partitions are only well defined with respect to highest weights. To each partition we associate a Young Tableau. There are $N-1$ rows, and the i^{th} row has ℓ_i columns. For example, the diagram



corresponds to the partition $\lambda = \{5,3,3,2,1\}$. We can also easily relate the diagram to the Dynkin labels; each λ_i tells you the number of columns with i boxes. Recall that a fundamental representation is a representation that has highest weight $\omega_\ell = (0,0,\cdots,0,1,0,\cdots,0)$, where the 1 is in the ℓ^{th} place. The young diagram is a vertical column of ℓ boxes;



We define the *transposed* tableau by exchanging rows and columns



The corresponding weight is written λ^t.

15. Semi-Standard Tableaux

In fact, tableaux in a modified form can describe every state in a given representation. The modified tableaux are called *semi-standard*. They consist of tableaux with an integer in each box, satisfying the relations detailed in figure 2. Thus,

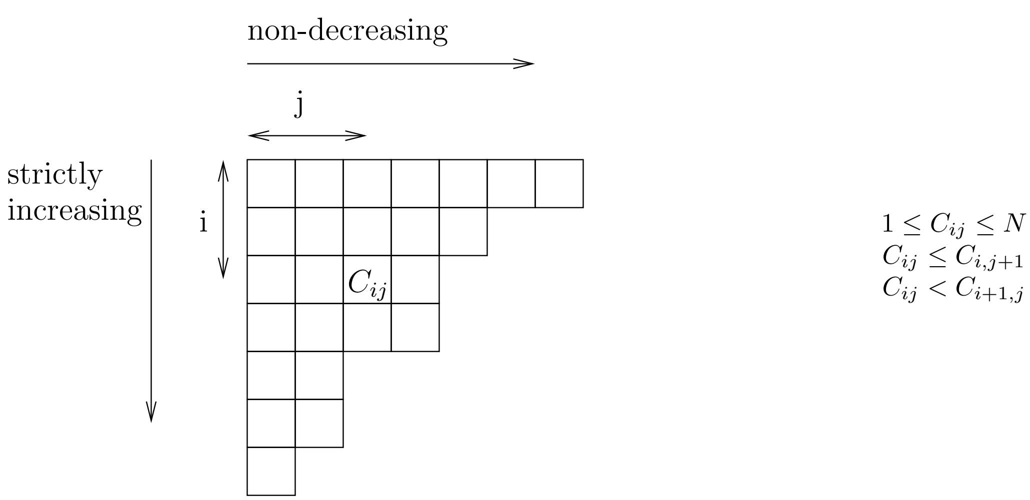


Figure 2. A schematic of the placement of integers in a tableau.

given a tableau of shape λ (i.e. specifying the representation), each semi-standard tableau corresponds to a state in L_λ. The numbering of the semi-standard tableau encodes the weight in a basis $\{\epsilon_i\}$, where $\epsilon_i = \omega_i - \omega_{i-1}$. As the algorithm may be understood without explicit mention of the origin of this basis, its real definition will be left out. For each box which contains an integer i, add a factor of $\epsilon_i = \omega_i - \omega_{i-1},\ i = 1, \cdots N$, where $\omega_0 = \omega_N = 0$.

For example, consider $\mathfrak{su}(3)$. There are two fundamental weights ω_1 and ω_2. In the representation corresponding to highest weight ω_1, we have possible semi-standard tableaux

$$\boxed{1}, \boxed{2}, \boxed{3}.$$

- The diagram $\boxed{1}$ means that $\epsilon_1 = \omega_1 - \omega_0$, and since $\omega_0 = 0$, in the fundamental basis the diagram is $(1, 0)$.
- The diagram $\boxed{2}$ means that $\epsilon_2 = \omega_2 - \omega_1$, so we see that in the fundamental basis we have $(-1, 1)$.
- The diagram $\boxed{3}$ means that $\epsilon_3 = \omega_3 - \omega_2$, and since $\omega_3 \equiv 0$, in the fundamental basis we will have $(0, -1)$.

Further insight may be found by the examination of the fundamental representation corresponding to ω_2. The diagram is two stacked boxes, ⊟, and as in the representation with highest weight ω_1 there are three possible semi-standard tableaux. Let us find the Dynkin weights of these states.

- The diagram $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$ implies that $\epsilon_1 = \omega_2 - \omega_1 + \omega_1 - \omega_0 = \omega_2$, and so in the fundamental basis, we have $(0, 1)$.
- The diagram $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}$ implies that $\epsilon_2 = \omega_3 - \omega_2 + \omega_1 - \omega_0 = \omega_1 - \omega_2$, and thus in the fundamental basis we have $(1, -1)$
- The diagram $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$ implies that $\epsilon_3 = \omega_3 - \omega_2 + \omega_2 - \omega_1 = -\omega_1$, which is $(-1, 0)$.

Let us finally consider the adjoint representation, described by the Young diagram ⊞. We shall work out a few of the diagrams and list the rest below.

- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$ We will have $\epsilon = 2(\omega_2 - \omega_1) + (\omega_1 - \omega_0) = 2\omega_2 - \omega_1$, or $(-1, 2)$.
- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ We will have $\epsilon = (\omega_3 - \omega_2) + (\omega_2 - \omega_1) + (\omega_1 - \omega_0) = 0$, or $(0, 0)$.

There are eight semi-standard tableaux possible:

[1 1 / 2]	[1 2 / 2]	[1 3 / 2]	[1 1 / 3]	[1 2 / 3]	[1 3 / 3]	[2 2 / 3]	[2 3 / 3]
$(1, 1)$	$(-1, 2)$	$(0, 0)$	$(2, -1)$	$(0, 0)$	$(1, -2)$	$(-2, 1)$	$(-1, -1)$

16. Characters

Another way to encode the content of a representation λ is by its *character* χ_λ;

$$\chi_\lambda = \sum_{\lambda' \in \Omega_\lambda} \mathrm{mult}_\lambda(\lambda') e^{\lambda'}.$$

This may also be rewritten in terms of a basis for the dual Cartan subalgebra;

$$\chi_\lambda = \sum_{\mu \in \mathfrak{h}^\vee} (\dim V_\mu) e^{\mu}.$$

In these expressions, e^λ and e^μ are formal exponentials satisfying

$$e^\lambda e^\mu = e^{\lambda + \mu}$$
$$e^\lambda(\xi) = e^{(\lambda, \xi)}$$

for $\xi \in \mathfrak{h}^\vee$. The character is then interpreted as a function on $\mathfrak{h}^\vee$;

$$\chi_\lambda(\xi) = \left[\sum_{\lambda' \in \Omega_\lambda} \mathrm{mult}_\lambda(\lambda') e^{\lambda'}\right](\xi) = \sum_{\lambda' \in \Omega_\lambda} \mathrm{mult}_\lambda(\lambda') e^{(\lambda', \xi)}$$

The character may also be written in terms of the Weyl vector ρ and a sum over the Weyl group.

$$\chi_\lambda = \frac{\sum_{w \in \mathcal{W}} \epsilon(w) e^{w(\lambda + \rho)}}{\sum_{w \in \mathcal{W}} \epsilon(w) e^{w\lambda}}$$

Here $\epsilon(w) = (-1)^{\ell(w)}$, and $\ell(w)$ is the *length* of w: $\ell(w)$ is the smallest integer k such that $w = s_1 \cdots s_k$.

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