

RIEMANN SURFACES AND THEIR MODULI

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ABSTRACT. The purpose of these notes is to give an overview of Riemann surfaces, and their moduli spaces. In general, theorems will be presented without proof, although references will be provided. These notes were originally prepared for the summer 2004 session of the UIUC Graduate String Seminar¹. Proofs of nearly all the theorems may be found, unless otherwise noted, in [FK80].

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1. MATHEMATICAL REVIEW

1.1. Complex structure. Let \mathcal{M} be a $2m$ -dimensional manifold, with an atlas $\{U_i, \phi_i\}$; $\bigcup_i U_i = \mathcal{M}$, $\phi_i: U_i \rightarrow \mathbb{C}^m$. If the functions $\phi_i \circ \phi_j^{-1}: \mathbb{C}^m \rightarrow \mathbb{C}^m$ are holomorphic on their domains of definition, the atlas is called complex-analytic. Two atlases $\{U_i, \phi_i\}$ and $\{V_i, \psi_i\}$ are said to be compatible if their union is again an atlas. Clearly this defines an equivalence relation on the set of atlases, an equivalence class of which is known as a *complex structure*. A manifold together with a complex structure is called a *complex manifold*. Its complex dimension is defined to be m ; $\dim_{\mathbb{C}} \mathcal{M} = m$. A complex manifold of complex dimension 1 is called a *Riemann surface*.

Another way to define a complex structure is to complexify the tangent bundle and introduce an almost-complex structure J . An almost-complex structure on \mathcal{M} is a tensor of type $(1,1)$ (a section of the bundle $\text{End}(T\mathcal{M} \otimes \mathbb{C}) \rightarrow \mathcal{M}$) which squares to -1 . In local coordinates this is

$$J_j^i J_k^j = -\delta_k^i.$$

An almost complex structure J is said to be integrable if the Nijenhuis tensor $N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$ vanishes for all smooth vector fields X and Y . An integrable almost-complex structure defines a complex structure on a manifold. In one complex dimension, it can be shown that the Nijenhuis tensor vanishes identically, so that every almost-complex structure defines a complex structure. Another useful fact is that any complex structure has a unique associated almost-complex structure.

A map of complex manifolds $f: M \rightarrow N$ is said to be holomorphic if it is holomorphic in any chart:

$$\phi_i^{(n)} \circ f \circ (\phi_j^{(m)})^{-1}: \mathbb{C}^m \rightarrow \mathbb{C}^n$$

Because transition functions between the sets of the cover are holomorphic, the holomorphicity of a map is independent of coordinates. Furthermore, this implies that every complex manifold is orientable. A map $f: \mathcal{M} \rightarrow \mathcal{N}$ is a *biholomorphism* if f is a holomorphic diffeomorphism (this implies that f^{-1} is holomorphic).

1.2. Conformal structure. Consider a Riemannian manifold \mathcal{M} , and define equivalence relation of metrics

$$g_1 \sim g_2 \quad \text{if } g_1 = e^{2\omega} g_2, \quad \omega \in C^\infty(\mathcal{M}).$$

This is known as *Weyl equivalence*, and an equivalence class of metrics is called a *conformal structure* on \mathcal{M} . Diffeomorphisms of \mathcal{M} which preserve the conformal structure are called *conformal transformations*.

A Riemannian manifold \mathcal{M} with a metric g is called *locally conformally flat* if every point $p \in \mathcal{M}$ has a coordinate neighbourhood U such that

$$g|_U = e^{2\omega} \sum_{i=1}^{2m} (dx^i)^2.$$

It turns out that every Riemann surface is locally conformally flat.

1.3. Equivalence of conformal and complex structure.

Theorem 1. *The set of conformal classes of metrics and the set of complex structures are in one-to-one canonical correspondence for Riemann surfaces.*

Specifying a complex structure completely specifies the conformal structure, and vice-versa. One consequence of this theorem is that the notion of a biholomorphism and a conformal transformation are equivalent.

2. CLASSIFICATION OF RIEMANN SURFACES

Mathematicians like to classify everything, up to isomorphism, and Riemann surfaces are no exception. Their classification is given by the *Uniformization theorem*. Before we begin to define the concepts we will need to state the Uniformization theorem, let us state a special case;

Theorem 2. Uniformization theorem for simply-connected ($\pi_1(\mathcal{M}) = \{e\}$) Riemann surfaces. *Up to conformal equivalence, there exist three simply-connected Riemann surfaces;*

- 1) $\hat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$ *the Riemann sphere*
- 2) \mathbb{C} *the complex plane*
- 3) $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ *the unit disk.*

In order to state the Uniformization theorem for arbitrary Riemann surfaces, we must first discuss covering maps, automorphism groups, and freely discontinuous group actions.

Definition 1. Automorphisms. An automorphism of a manifold \mathcal{M} is an biholomorphism $f: \mathcal{M} \rightarrow \mathcal{M}$. The set of automorphisms is denoted $\text{Aut}(\mathcal{M})$ and forms a group under composition.

Definition 2. Universal cover. Let \mathcal{M} be any connected Riemann surface. The *universal cover* of \mathcal{M} is a simply-connected Riemann surface Σ , equipped with a surjective map $\pi: \Sigma \rightarrow \mathcal{M}$. In addition, every point $p \in \mathcal{M}$ has a neighbourhood U where $\pi^{-1}(U)$ (the pre-image, *not* the inverse) is a countable disjoint union of sheets (see figure 2.1

$$\pi^{-1}(U) = \bigcup_{\alpha \in \mathbb{Z}} V_\alpha.$$

Each V_α (a sheet) is homeomorphic to U via π .

Definition 3. Covering groups. If $\pi: \Sigma \rightarrow \mathcal{M}$ is a covering map, then there is a group \mathcal{G} of homeomorphisms of Σ such that the quotient space Σ/\mathcal{G} is isomorphic to \mathcal{M} . \mathcal{G} is called the covering group, and is isomorphic to $\pi_1(\mathcal{M})$.

One may also inquire as to when a group \mathcal{G} of homeomorphisms gives rise to a surface Σ/\mathcal{G} with projection map equal to the quotient map. The condition is that of free discontinuous action.

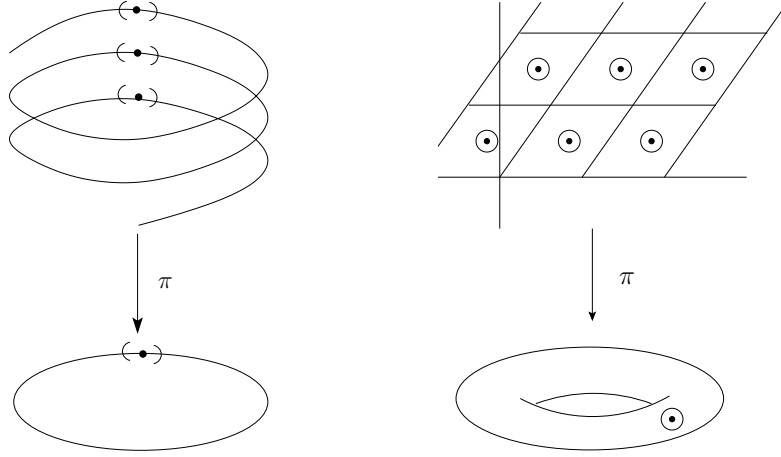


FIGURE 2.1. A cartoon of two spaces, S^1 and T^2 , with their universal covers \mathbb{R} and \mathbb{C} and projection maps $\pi: \mathbb{R} \rightarrow S^1$ and $\pi: \mathbb{C} \rightarrow T^2$.

Definition 4. A group \mathcal{G} acts freely discontinuously at a point $p \in \Sigma$ if there exists a neighbourhood U of p such that

$$gU \cap U = \emptyset \quad \forall \quad g \in \mathcal{G} - \{e\}.$$

If the action is freely discontinuous at every point, the quotient space Σ/\mathcal{G} is a manifold.

If \mathcal{M} is a Riemann surface and Σ is its universal cover, the complex structure pulls back via the projection map π and the covering group consists of conformal automorphisms; it is a subgroup of $\text{Aut}(\Sigma)$.

2.1. Uniformization theorem for Riemann surfaces. Now we are ready to state the Uniformization theorem for arbitrary Riemann surfaces.

Theorem 3. Every Riemann surface \mathcal{M} is conformally equivalent to Σ/\mathcal{G} , where

$$\Sigma = \begin{cases} \hat{\mathbb{C}} \\ \mathbb{C} \\ \Delta \end{cases}$$

is the universal cover of \mathcal{M} , and \mathcal{G} is a subgroup of $\text{Aut}(\Sigma)$ admitting a freely discontinuous action on \mathcal{M} . In addition, $\mathcal{G} \simeq \pi_1(\mathcal{M})$.

2.2. Automorphisms. Subgroups of the group of automorphisms of the three Riemann surfaces play an important rôle in Theorem 3, so it is a good idea to know what Aut is for each surface.

2.2.1. $\text{Aut}(\hat{\mathbb{C}})$. To find $\text{Aut}(\hat{\mathbb{C}})$, we use the fact that

$$\hat{\mathbb{C}} \simeq \mathbb{C}\mathbb{P}^1 \equiv \mathbb{P}^1,$$

with the isomorphism given in homogeneous coordinates on \mathbb{P}^1 by

$$(1) \quad ([z_1, z_2]) \mapsto \frac{z_1}{z_2}, \quad ([1, 0]) \equiv \{\infty\}$$

The action of $\text{GL}(2, \mathbb{C})$ on \mathbb{C}^2 projects to an action of $\text{PL}(2, \mathbb{C}) \equiv \{\text{GL}(2, \mathbb{C})/\lambda, \lambda \in \mathbb{C}^*\}$ on \mathbb{P}^1 . Then, $\text{PL}(2, \mathbb{C})$ is the group $\text{Aut}(\hat{\mathbb{C}})$, whose action on $\hat{\mathbb{C}}$ is

$$(2) \quad \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto z' = \frac{az + b}{cz + d},$$

by virtue of equation (1). Such a transformation [the RHS of (2)] is called a Möbius transformation.

$$\text{Aut}(\hat{\mathbb{C}}) \simeq \text{PL}(2, \mathbb{C}).$$

2.2.2. $Aut(\mathbb{C})$. To find $Aut(\mathbb{C})$, we note that the conformal automorphisms of \mathbb{C} will be those automorphisms of $\hat{\mathbb{C}}$ which fix the point ∞ . It is clear that a Möbius transformation which fixes ∞ must have $c = 0$;

$$z' = az + b, \quad a \in \mathbb{C}^*, b \in \mathbb{C}$$

This group is known as $Aff(1, \mathbb{C})$, the affine transformations of the plane. It is isomorphic to the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Thus,

$$Aut(\mathbb{C}) \simeq Aff(1, \mathbb{C}).$$

2.2.3. $Aut(\Delta)$. The automorphisms of Δ do not have such a straightforward derivation. An overview is presented: The unit disk is conformally equivalent to the upper-half plane $\mathcal{U} = \{z \in \mathbb{C} | \Im(z) > 0\}$ (Δ and \mathcal{U} will be used interchangeably). The group of automorphisms of \mathcal{U} is $PL(2, \mathbb{R})$, so

$$Aut(\Delta) \simeq PL(2, \mathbb{R}).$$

2.3. The Exceptional Riemann Surfaces.

Theorem 4. *With three classes of exceptions, the fundamental group π_1 of a Riemann surface is non-abelian.*

- (1) *The only surfaces up to conformal equivalence with trivial fundamental group are \mathbb{C} , $\hat{\mathbb{C}}$, and Δ .*
- (2) *The only surfaces (u.t.c.e.) with $\pi_1 \simeq \mathbb{Z}$ are \mathbb{C}^* , Δ^* , and $\Delta_r = \{z \in \mathbb{C} | r < |z| < 1\}$.*
- (3) *The only surfaces (u.t.c.e.) with $\pi_1 \simeq \mathbb{Z} \oplus \mathbb{Z}$ are tori $\mathbb{C}/\Lambda(\omega, \eta)$, Λ a lattice (see equation 3) in \mathbb{C} .*

These conformal classes of surfaces are known as exceptional.

3. MODULI OF RIEMANN SURFACES

Two Riemann surfaces can have the same underlying topological space, and yet be conformally inequivalent (have different complex structures). The set of conformally-inequivalent surfaces over the same topological space is known as a moduli space. Is there structure on moduli spaces? The answer to this question is yes, generically, and the structure can be very interesting. The general solution is complicated, so we will be content with examining a few important examples.

To aid our study, we shall make use of the following theorem.

Theorem 5. *Two Riemann surfaces have the same universal cover Σ and covering groups conjugate in $Aut(\Sigma)$ if and only if they are conformally equivalent. That is, for $G_1, G_2 \subset Aut(\Sigma)$, there exists a $g \in Aut(\Sigma)$ such that $gG_1g^{-1} = G_2$.*

3.1. **Surfaces with universal cover $\hat{\mathbb{C}}$.** As was found before, $Aut(\hat{\mathbb{C}})$ is the Lie group $PL(2, \mathbb{C})$. Recall that if we think of $\hat{\mathbb{C}}$ as $\mathbb{C} \cup \infty$, then the action of $Aut(\hat{\mathbb{C}})$ is that of a Möbius transformation, equation (2).

Proposition 1. *The only Riemann surface with universal cover $\hat{\mathbb{C}}$ is $\hat{\mathbb{C}}$ itself.*

Proof. First we show that Möbius transformations fix at least one point of $\hat{\mathbb{C}}$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PL(2, \mathbb{C})$. Then, a fixed point satisfies $z = \frac{az + b}{cz + d}$, which simplifies to

$$\begin{aligned} z &= \frac{a-d}{2c} \pm \frac{1}{2c} \sqrt{(d-a)^2 - 4cb} & c \neq 0 \\ z &= \frac{b}{d-a} & c = 0, a \neq d \\ z &= \infty & c = 0, a = d, b \neq 0. \end{aligned}$$

These equations clearly have solutions for any element of $PL(2, \mathbb{C})$. Thus, we see that every element of $Aut(\hat{\mathbb{C}})$ fixes at least one point of $\hat{\mathbb{C}}$, and since no proper subgroup of $PL(2, \mathbb{C})$ can act freely discontinuously on $\hat{\mathbb{C}}$, we have the desired result. \square

An obvious consequence of this proposition is that the moduli space of genus zero surfaces is a one-point set. In fact, all three of the simply-connected surfaces have one-point moduli spaces.

3.2. **Surfaces with universal cover \mathbb{C} .** Recall that $\text{Aut}(\mathbb{C}) = \text{Aff}(1, \mathbb{C})$, with $z \mapsto az + b$. We will make use of the fact that

Theorem 6. *If the (holomorphic) universal covering space of \mathcal{M} is \mathbb{C} , then \mathcal{M} is conformally equivalent to \mathbb{C} , \mathbb{C}^* , or T^2 , a torus.*

For a proof, see [Ahl79]. The respective covering groups are $\{e\}$, \mathbb{Z} , and $\mathbb{Z} \oplus \mathbb{Z}$.

Examine first the case $\mathcal{G} = \mathbb{Z}$. We can take $z \mapsto z + 1$ as a generator, so that the covering map $\pi: \mathbb{C} \rightarrow \mathbb{C}^*$ is just the exponential map $\pi(z) = \exp(2\pi\sqrt{-1}z)$. The moduli space for \mathbb{C}^* is again trivial.

$\mathcal{G} = \mathbb{Z} \oplus \mathbb{Z}$ is a bit more complicated. We shall make use of Theorem 5. Consider a lattice in \mathbb{C} ;

$$(3) \quad \Lambda(\omega, \eta) = \{m\omega + n\eta \mid m, n \in \mathbb{Z}; \omega, \eta \in \mathbb{C}^* \text{ linearly independent}\}.$$

Clearly it is a discrete group, isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and the quotient $\mathbb{C}/\Lambda(\omega, \eta)$ is a torus (see figure 2.1). When do two lattices give conformally equivalent tori? By Theorem 5, the two lattices must be conjugate in $\text{Aut}(\mathbb{C})$. Before we establish conditions for the conjugacy of two lattices, we elucidate one further complication. Two lattices $\Lambda(\omega_1, \eta_1)$ and $\Lambda(\omega_2, \eta_2)$ may be identical; that is, they are composed of the same set of points in \mathbb{C} .

Lemma 1. *The two lattices define the same set of points (see figure 3.1) if the two basis elements of one lattice are linear combinations of the other two (with integer coefficients);*

$$\begin{pmatrix} \omega'_1 \\ \eta'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \eta_2 \end{pmatrix},$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $GL(2, \mathbb{Z})$. Furthermore, multiplying the entire matrix by any integer will yield the same lattice; we want $PL(2, \mathbb{Z})$.

Proof. The proof is easy, but involves much typing. Thus it is left to the interested reader. □

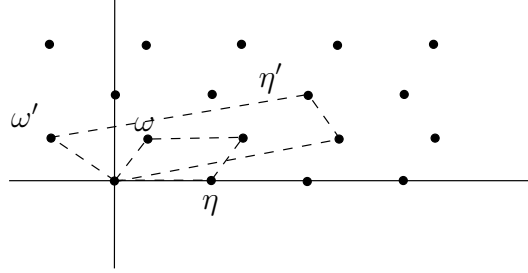


FIGURE 3.1. Two bases defining identical lattices in \mathbb{C} .

Proposition 2. *The conjugacy class of $\Lambda(\omega, \eta)$ in $\text{Aut}(\mathbb{C})$ is the set of lattices of the form $\Lambda(a\omega, a\eta)$, with $a \in \mathbb{C}^*$.*

Proof. An element $(a, b) \in \text{Aut}(\mathbb{C})$ acts on a generator $h_\omega: z \mapsto z + \omega$ of the covering group as $(a, b) \cdot h_\omega(z) = a(z + \omega) + b$, and thus

$$(a, b)h_\omega(a, b)^{-1}(z) = z + a\omega.$$

□

Now, we define $\tau = \eta/\omega$ and without loss of generality, choose $\Im(\tau) > 0$ (i.e. since any two points in the lattice define it, choose them appropriately: see figure 3.1). Furthermore, we choose $a = 1/\omega$, so that every lattice is conjugate to one of the form $\Lambda(1, \tau)$. Combining this result and that of Lemma 1, we see that a two lattices are equivalent if they are related by the Möbius transformation

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PL(2, \mathbb{Z})$.

Theorem 7. *The moduli space of the torus is $\mathcal{U}/\mathrm{PL}(2, \mathbb{Z})$.*

3.3. **Surfaces with universal cover \mathcal{U} .** Except for the four surfaces mentioned previously ($\hat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*$, and T^2), all Riemann surfaces have \mathcal{U} as their universal covering space. There are many subgroups of $\mathrm{Aut}(\mathcal{U})$ which act freely discontinuously, the fixed-point-free *Fuchsian groups*. Rather than beginning a long discourse, we shall leave them for another time. For now, we will find the moduli spaces of the rest of the exceptional Riemann surfaces, Δ, Δ^* , and Δ_r .

To keep this lecture under an hour, we simply present the results and refer the reader to §IV.7.2 of [FK80].

Surface	π_1	Moduli space
$\hat{\mathbb{C}}$	$\{e\}$	$\{\mathrm{pt}\}$
\mathbb{C}	$\{e\}$	$\{\mathrm{pt}\}$
\mathbb{C}^*	\mathbb{Z}	$\{\mathrm{pt}\}$
T^2	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathcal{U}/\mathrm{PL}(2, \mathbb{Z})$
Δ	$\{e\}$	$\{\mathrm{pt}\}$
Δ^*	\mathbb{Z}	$\{\mathrm{pt}\}$
Δ_r	\mathbb{Z}	$(1, \infty)$

REFERENCES

- [Ahl79] Lars V. Ahlfors. *Complex Analysis*. McGraw-Hill Book Company, 3rd edition, 1979.
 [FK80] Hershel M. Farkas and Irwin Kra. *Riemann Surfaces*. Springer-Verlag New York, 1st edition, 1980.