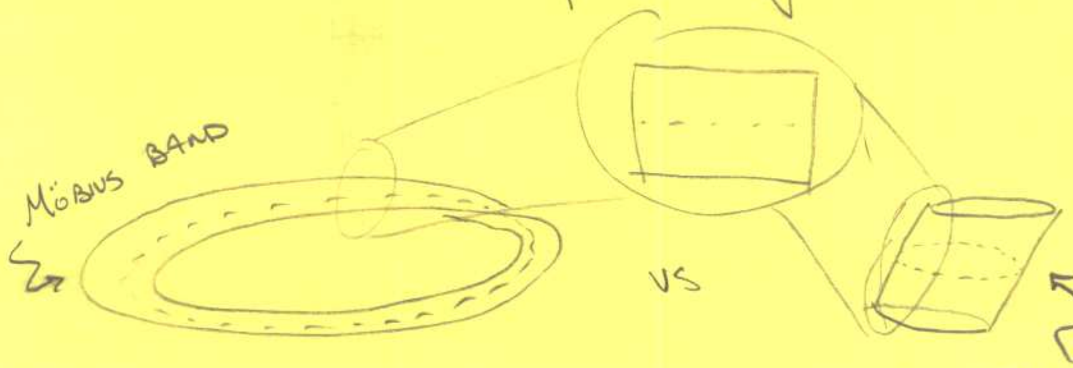
BUNDLES.

WE WISH TO CONSIDER SPACES WHICH ARE LOCALLY, BUT NOT NECESSARILY GLOBALLY, PRODUCTS.



BOTH APPEAR LOCALLY TO BE THE PRODUCT OF THE CIRCLE & THE REAL LINE, BUT GLOBALLY THEY ARE VERY DIFFERENT

CONSIDER A MANIFOLD $\mathcal{M}$, WITH COÖRD. NHBDS $\{\mathcal{U}_i, \varphi_i\}$
A BUNDLE ON $\mathcal{M}$ CONSISTS OF
(i) A MANIFOLD $\mathcal{F}$, THE TYPICAL FIBRE.
(ii) A LIE GROUP $\mathcal{G}$, ACTING ON $\mathcal{F}$ FROM THE LEFT
(iii) WHENEVER $\mathcal{U}_i \cap \mathcal{U}_j \neq \phi$, A MAP (SMOOTH)

$$t_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \longrightarrow \mathcal{G}$$

(iv) A DIFFERENTIABLE MANIFOLD $\mathcal{E}$, THE TOTAL SPACE OF THE BUNDLE, & A MAP

$$\pi : \mathcal{E} \to \mathcal{M}$$

CALLED THE PROJECTION MAP. FOR ALL $p \in \mathcal{M}$, $\pi^{-1}(p) \simeq \mathcal{F}$.

THE Hopf BUNDLE

CONSIDER $S^3 \subset \mathbb{R}^4$; $\{x_i \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$

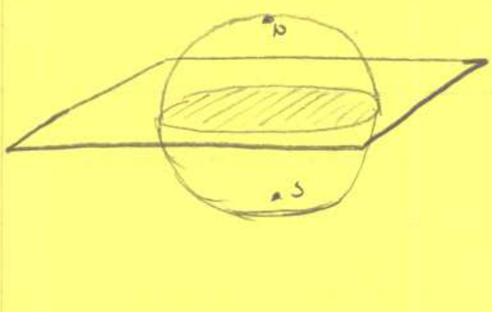
DEFINE A MAP $\pi : S^3 \to S^2 \subset \mathbb{R}^3$ AS FOLLOWS

$$y_1 \equiv y_1 \circ \pi = 2(x_1 x_3 + x_2 x_4)$$
$$y_2 \equiv y_2 \circ \pi = 2(x_2 x_3 - x_1 x_4)$$
$$y_3 \equiv y_3 \circ \pi = x_1^2 + x_2^2 - x_3^2 - x_4^2$$

ONE CAN CHECK THAT $y_1^2 + y_2^2 + y_3^2 = 1$, SO $\pi \to S^2$
WE CAN ALSO DEFINE LOCAL COÖRD'S ON THE 2-SPHERE;

$$z^0 = x^1 + i x^2 \qquad z^1 = x^3 + i x^4$$

& ON THE NORTHERN HEMISPHERE $Z = \frac{z^0}{z^1}$ IS A VALID COÖRDINATE, WHILE ON THE SOUTHERN, $W = z^1/z^0$ IS.
THIS IS JUST "POLAR PROJECTION".



FINALLY, NOTE Z & W ARE INVARIANT UNDER $(z^0, z^1) \mapsto \lambda(z^0, z^1)$ $\forall\ \lambda \in U(1)$ (AND SO ARE THE y_i!)

THUS, WE CAN MAKE A "$U(1)$" TRANSFORMATION ON THE x'S WHICH LEAVES THE y'S INVARIANT; THIS TELLS US THAT IN FACT, S^3 IS A $U(1)$ BUNDLE OVER S^2!

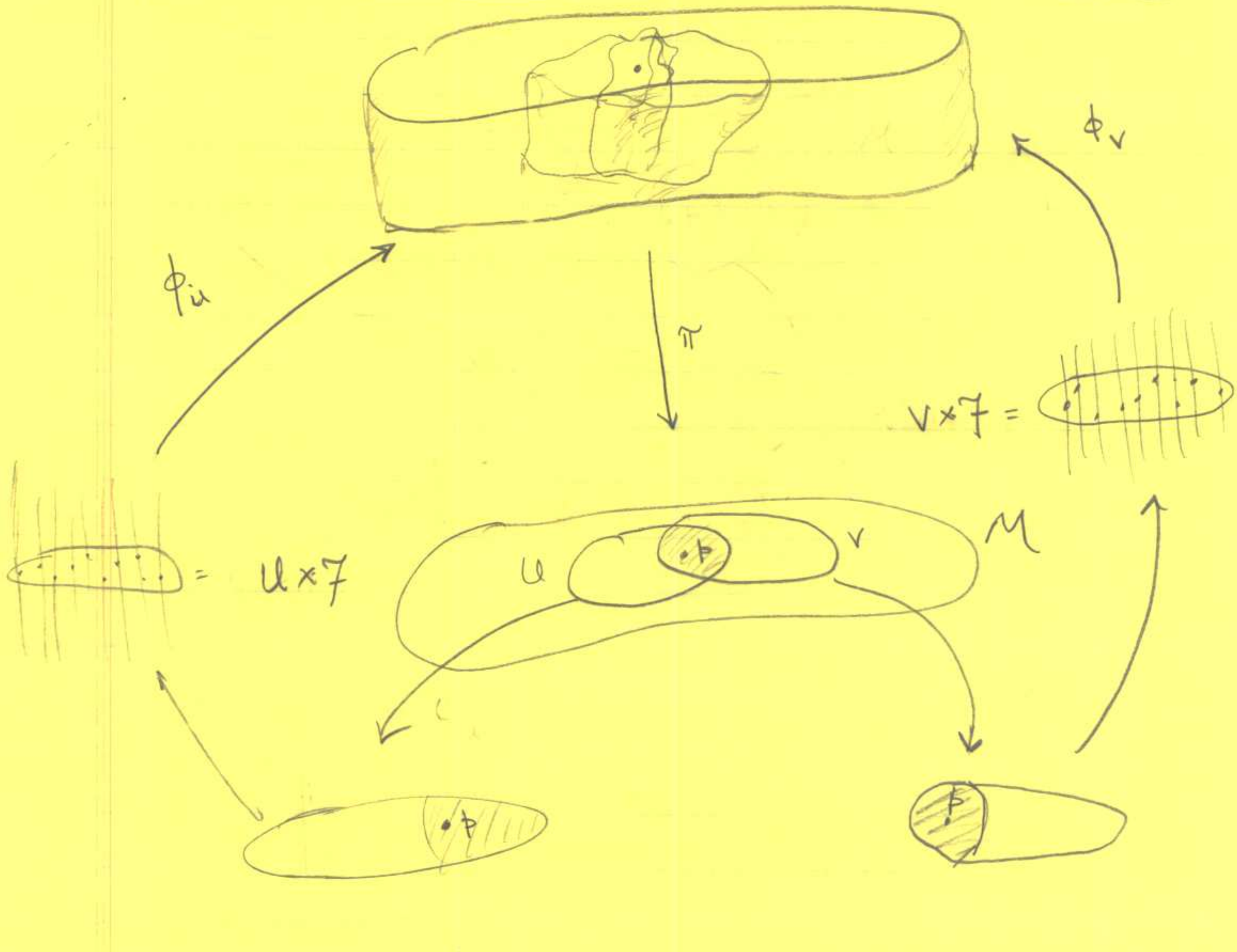
②

ADDITIONALLY, for EACH COÖRDINATE NEIGHBORHOOD $\mathcal{U}_i \subset \mathcal{M}$,

(v) A DIFFEOMORPHISM $\phi_i : \mathcal{U}_i \times \mathcal{F} \to \pi^{-1}(\mathcal{U}_i) \subset \mathcal{E}$ SATISFYING

$$\pi \circ \phi_i(p, f) = p$$

& $\quad t_{ij}(p) = \phi_i^{-1} \circ \phi_j : \mathcal{F} \to \mathcal{F}$



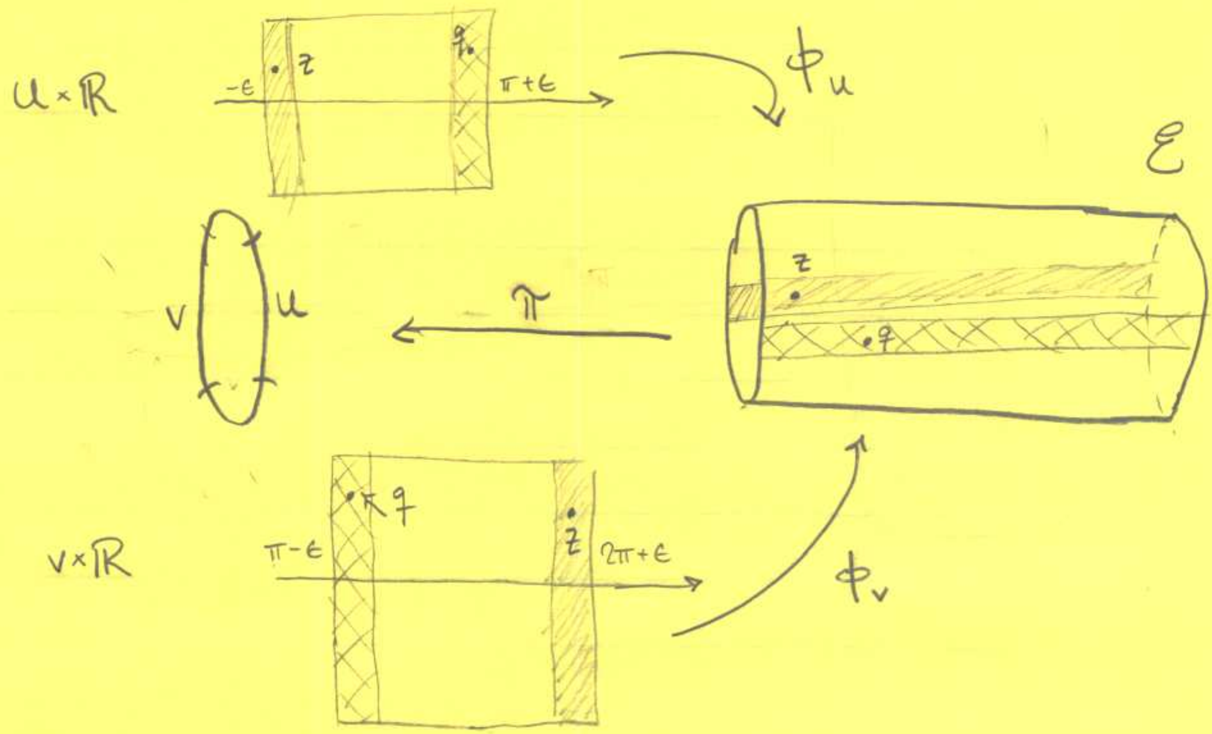
③

Cylinder example; $\mathcal{F} = \mathbb{R}$

$\mathcal{G} = \mathbb{R}^*$

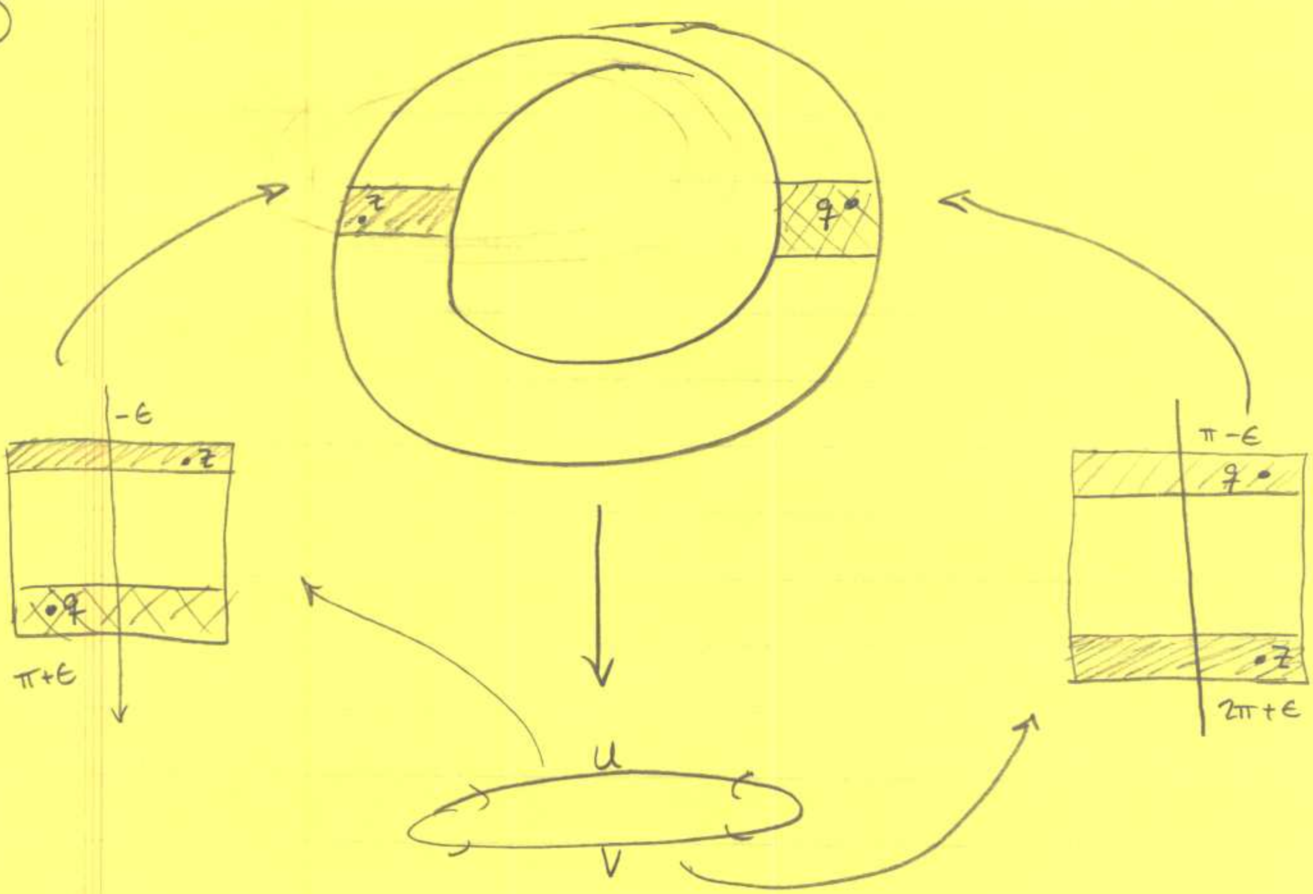
$M = S^1$, $\quad U = (-\epsilon, \pi+\epsilon) \quad V = (\pi-\epsilon, 2\pi+\epsilon)$

$\mathcal{E}$ = cylinder



on $U \cap V$, $\quad t_{UV} = 1$

④



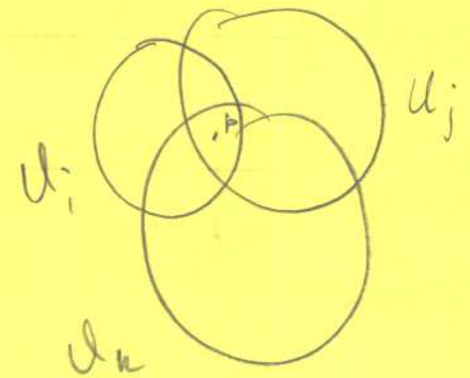
ON $U \cap V$, $t_{UV} = \begin{cases} -1 & \theta > 3\pi/2 \\ 1 & \theta < 3\pi/2 \end{cases}$

THE TRANSITION FUNCTIONS MUST SATISFY CONSISTENCY CONDITIONS,

$$t_{ij}(p) = t_{ji}^{-1}(p) \quad \forall \ p \in \mathcal{U}_i \cap \mathcal{U}_j$$

$$t_{ij}(p) \cdot t_{jk}(p) = t_{ik}(p) \quad \forall \ p \in \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$$

⑤



ANOTHER VIEWPOINT

TAKE $\quad \bigcup_i \mathcal{U}_i \times F \big/ \sim \;\equiv\; \mathcal{E}$

WHERE for $(p_i, f_i) \in \mathcal{U}_i \times F$
$\quad$ & $(p_j, f_j) \in \mathcal{U}_j \times F$,

$$(p_i, f_i) \sim (p_j, f_j) \text{ if } p_i = p_j \;\&\; t_{ij}(p)\cdot f_i = f_j$$

THEN $\quad \pi : [(p, f)] \equiv p$
$\quad$ & $\phi_i : (p, f) \longmapsto [(p, f)]$

WE WILL UTILIZE THIS VIEWPOINT WHEN CONSTRUCTING ASSOCIATED VECTOR BUNDLES.

(6)

A VECTOR BUNDLE IS A BUNDLE WHOSE FIBRE IS A VECTOR SPACE. WE WILL TAKE

$$\mathcal{F} = \mathbb{R}^n \quad \text{OR} \quad \mathcal{F} = \mathbb{C}^n$$

FOR THESE FIBRES, THE STRUCTURE GROUP WILL ALWAYS BE

$$\mathcal{G} \subseteq GL(n,\mathbb{R}) \quad \text{OR} \quad \mathcal{G} \subseteq GL(n,\mathbb{C})$$

EXAMPLE; LAST TIME, WE DISCUSSED THE TANGENT SPACE AT A POINT T_pM. IF WE SET $\mathcal{F} = T_pM$, & TAKE THE TRANSITION FUNCTIONS TO BE

$$t_{ij}(p) = \frac{\partial x^{\mu}}{\partial y^{\nu}}(p)$$

FOR COORDINATES y^{ν} ON U_i & x^{μ} ON U_j, WE HAVE THE TANGENT BUNDLE, TM.

THE DIMENSION OF THE FIBRE IS CALLED THE RANK OF THE VECTOR BUNDLE. NOTE, $\text{RANK}(TM) = \dim M = m$. THE DIMENSION OF THE TOTAL SPACE OF THE BUNDLE IS THE SUM OF THE RANK & THE DIMENSION OF THE BASE.

$$\begin{aligned} \dim TM &= \text{RANK}(TM) + \dim M \\ &= 2m \end{aligned}$$

7

A rank one vector bundle has the special name of line bundle: $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$, & the structure group $\mathcal{G} \subset GL(1, \mathbb{R})$ consists of non-zero functions.

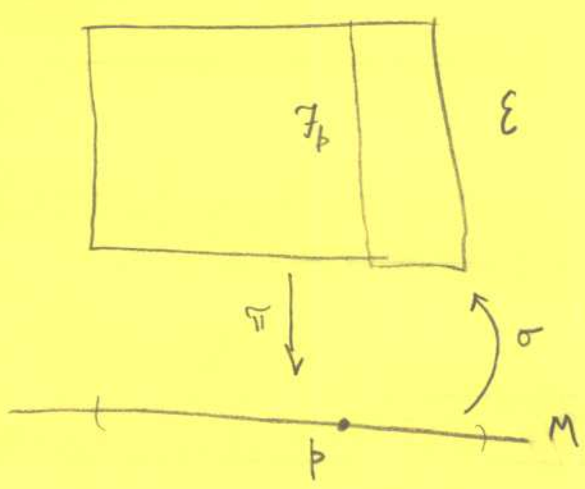
Sections of Bundles

We've seen there is a canonical projection map from the total space to the base space

$$\begin{array}{c} \mathcal{E} \\ \downarrow \pi \\ \mathcal{M} \end{array}$$

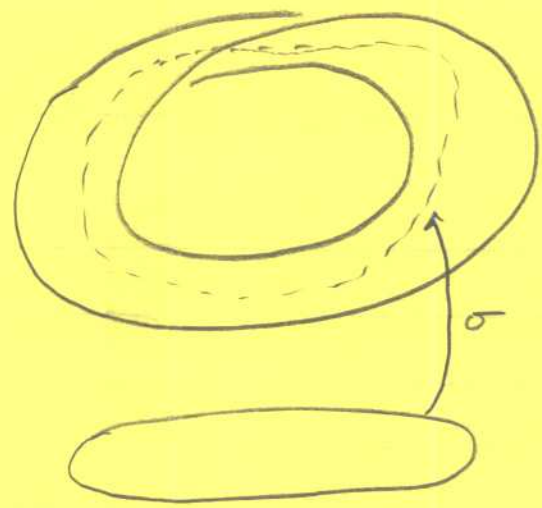
A section of the bundle $\mathcal{E}$ is a map $\sigma : \mathcal{M} \rightarrow \mathcal{E}$, which is compatible with the projection map

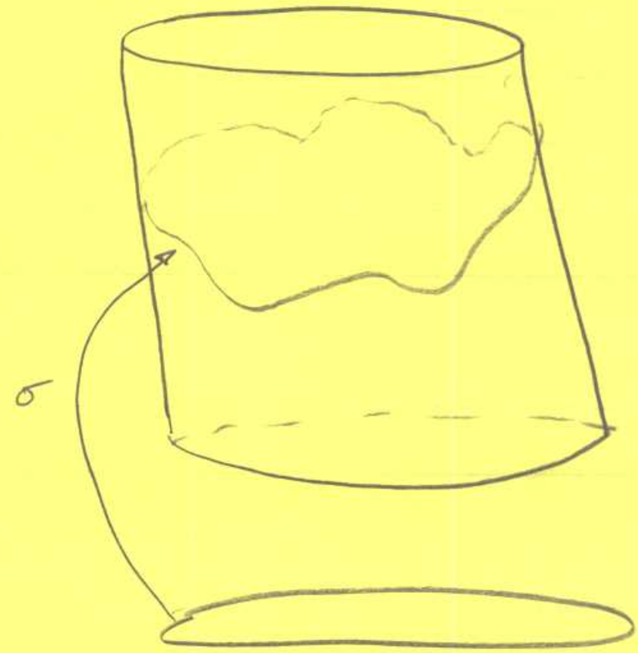
$$\pi \circ \sigma(p) = p$$



$$\mathcal{F}_p = \pi^{-1}(p)$$

$$\sigma(p) \in \mathcal{F}_p \quad \forall\, p$$





FOR VECTOR BUNDLES, THERE ALWAYS EXISTS AT LEAST ONE SECTION, THE 'ZERO SECTION'.

LOCALLY, ON SOME $U_i \times \mathbb{R}^n$, $\sigma(p) = 0 \quad \forall p \in U_i$ MAKES SENSE, & SINCE t_{ij} IS A MATRIX $\forall j$, $t_{ij} \cdot \sigma = 0$.

IN FACT, A LOCAL SECTION, THAT IS, A SECTION OF

$$\begin{array}{c} \mathcal{U}_i \times \mathbb{R}^n \\ \downarrow \\ \mathcal{U}_i \end{array},$$

IS SIMPLY THE (SMOOTH) ASSIGNMENT OF A VECTOR TO EACH POINT OF $\mathcal{U}_i$. CONSIDER THE TANGENT BUNDLE; SECTIONS TRANSFORM USING THE TRANSITION FUNCT'S $\frac{\partial x^\mu}{\partial y^\nu}$. IF $\sigma_i^\mu(p) : \mathcal{U}_i \to \mathcal{U}_i \times T_pM$, ON $\mathcal{U}_j \cap \mathcal{U}_i$ WE HAVE

$$\sigma_j^\nu(p) = \frac{\partial x^\nu}{\partial y^\mu} \sigma_i^\mu(p)$$

BUT, THIS IS EXACTLY HOW VECTOR FIELDS TRANSFORM. THUS,

$$\Gamma(M, TM) \cong \mathfrak{X}(M)$$

SPACE OF SECTIONS

RECALL THAT $\mathfrak{X}(M)$ FORMS A VECTOR SPACE; $\Gamma(M, TM)$ MUST BE AS WELL. INDEED THIS IS TRUE FOR THE SPACE OF SECTIONS OF ANY LINE BUNDLE.

$$(\sigma + \tau)(p) \equiv \sigma(p) + \tau(p)$$

+ IN Γ + IN $\mathbb{R}^n$