# Bundles II

Last week, we were introduced to bundles, specifically to vector bundles. Today, we will examine them further, & explore their relationship with principle bundles.

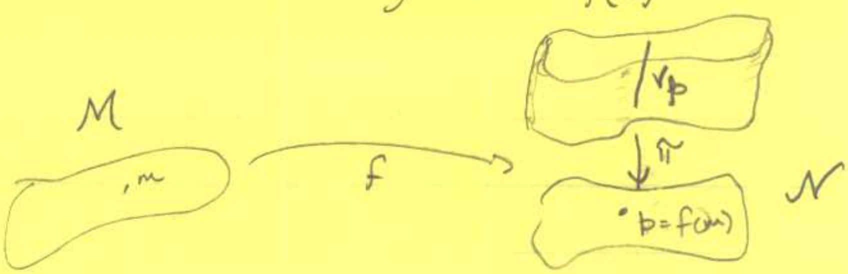
First though, we examine how a bundle

$$\begin{array}{c}\mathcal{E}\\ \downarrow \pi \\ \mathcal{N}\end{array}$$

and a smooth map $f: \mathcal{M} \to \mathcal{N}$ induces a bundle on $\mathcal{M}$. If $V_p$ is the fibre of $\mathcal{E}$ at $p$,

$$V_p = \pi^{-1}(p) \subset \mathcal{E},$$

the new bundle has fibre $V_{f(m)}$. That is

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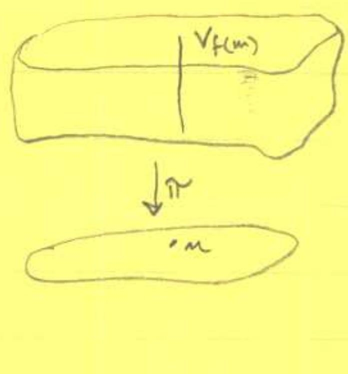

THE NEW BUNDLE IS CALLED THE PULLBACK of $\mathcal{E}$ BY $f$, & IS DENOTED

$$f^*\mathcal{E}.$$

IT IS CONSTRUCTED AS follows. LET $\{\mathcal{U}_i\}$ BE A COÖRDINATE COVER of $N$. SINCE $f$ IS SMOOTH, & THUS CTS, WE WILL HAVE THAT

$$\{f^{-1}(\mathcal{U}_i)\}$$

IS A COÖRDINATE COVER for $M$. For EACH $i$, WE WRITE

$$f^{-1}(\mathcal{U}_i)\times V \xrightarrow{\phi_i} f^*\mathcal{E}$$

AS THE LOCAL TRIVIALIZATION, AND WHEN $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$,

$$t^*_{ij}(p) = t_{ij}(f(p))$$

ARE THE TRANSITION functions. IN TERMS of THE Definition of THE BUNDLE AS A COLLECTION of EQUIVALENCE CLASSES, WE HAVE for $(p_i, v_i) \in f^{-1}(\mathcal{U}_i)\times V$ & $(p_j, v_j) \in f^{-1}(\mathcal{U}_j)\times V$,

$$(p_i, v_i) \sim (p_j, v_j) \quad \text{if} \quad p_i = p_j \in M$$
$$\& \quad v_i = t^*_{ij}\, v_j$$

③

PULLBACKS ALSO INDUCE BUNDLE MAPS; IF

$$\begin{array}{ccc} \mathcal{E} & & \mathcal{F} \\ \pi_{\mathcal{E}} \downarrow & \& & \downarrow \pi_{\mathcal{F}} \\ N & & M \end{array}$$ ARE TWO BUNDLES,

A BUNDLE MAP (OR MORPHISM) IS A SMOOTH MAP OF THE TOTAL SPACES, WHICH IS COMPATIBLE WITH THE PROJECTION MAP;

$\exists$ A $q \in M$, SUCH THAT $\forall$ $v \in \mathcal{E}_p$ for SOME $p \in N$, $f(v) \in \mathcal{F}_q$

$f$ THUS <u>INDUCES</u> A MAP $f': N \to M$ AS

$$f'(p) = \pi_{\mathcal{F}} f(\underbrace{0(p)}_{\text{ZERO SECTION}})$$

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{F} \\ \pi_{\mathcal{E}} \downarrow & & \downarrow \pi_{\mathcal{F}} \\ N & \xrightarrow{f'} & M \end{array}.$$

THEN for $g': N \to M$, THERE IS A BUNDLE MAP $g: g'^{*}\mathcal{E} \to \mathcal{E}$ CONSTRUCTED IN THE OBVIOUS WAY;

$$\begin{array}{ccc} g'^{*}\mathcal{E} & \xrightarrow{g} & \mathcal{E} \\ \downarrow & & \downarrow \\ N & \xrightarrow[g']{} & M \end{array}$$

GIVEN VECTOR SPACES $V$ & $W$, WE FORM THEIR DIRECT SUM BY SPECIFYING ITS BASIS IN TERMS OF BASES FOR $V$ & $W$;

$$V \oplus W = \mathrm{Span}\{v_1, \cdots, v_r, w_1, \cdots, w_s\}$$

WITH $\{v_i\}$ A BASIS FOR $V$ & $\{w_j\}$ FOR $W$. SIMILARLY, WE FORM THEIR TENSOR PRODUCT BY

$$V \otimes W = \mathrm{Span}\{v_1 \otimes w_1, v_1 \otimes w_2, \cdots, v_1 \otimes w_s, \cdots, v_r \otimes w_s\},$$

WHERE THE NOTATION $v_i \otimes w_j$ MEANS AN OBJECT SATISFYING

(i) $v_i \otimes w_j + v_k \otimes w_j = (v_i + v_j) \otimes w_j$
(ii) $v_i \otimes w_j + v_i \otimes w_k = v_i \otimes (rw_j + w_k)$
(iii) $a(v_i \otimes w_j) = (a v_i) \otimes w_j = v_i \otimes (a w_j)$

GIVEN TWO VECTOR BUNDLES $\mathcal{E}$ AND $\mathcal{F}$, WITH TYPICAL FIBRES $V$ & $W$, WE CAN FORM THE SUM BUNDLE OR WHITNEY SUM BUNDLE AS FOLLOWS

WE HAVE $\begin{array}{c}\mathcal{E}\\ \downarrow\\ M\end{array}$ & $\begin{array}{c}\mathcal{F}\\ \downarrow\\ M\end{array}$. WE CAN CONSTRUCT

$$\begin{array}{c}\mathcal{E}\times\mathcal{F}\\ \downarrow\\ M\times M\end{array}$$

WITH TYPICAL FIBRE $V\times W$, & TRANSITION FUNCTIONS ON

$$\mathcal{U}_i\times\mathcal{U}_j \cap \mathcal{U}_k\times\mathcal{U}_l \neq \emptyset$$

GIVEN BY $\begin{pmatrix} t_{ik} & 0 \\ 0 & u_{jl}\end{pmatrix}$. $\mathcal{E}$ TRANS$^n$ FCT$^n$, $\mathcal{F}$ TRANS$^n$ FCT$^n$

THEN, THE SUM BUNDLE $\mathcal{E}\oplus\mathcal{F}$ IS THE PULLBACK OF $\mathcal{E}\times\mathcal{F}$ BY THE DIAGONAL MAP

$$\Delta: M \to M\times M$$
$$\Delta: p \mapsto (p,p)$$

BASICALLY, THIS IS THE BUNDLE WHOSE FIBRE AT $p$ IS $V_p\times W_p$, & WHOSE TRANSITION FUNCTIONS ARE

$$\begin{pmatrix} t_{ij}(p) & 0 \\ 0 & u_{ij}(p)\end{pmatrix}$$

THE BUNDLE IS CONSTRUCTED IN THIS WAY BECAUSE IT IS INTRINSIC; FOR TOTAL SPACES $\mathcal{E}$ & $\mathcal{F}$, WE HAVE A NATURAL CONSTRUCTION OF $\mathcal{E}\times\mathcal{F} \to M\times M$, & THE NATURAL DIAGONAL MAP $M \xrightarrow{\Delta} M\times M$.

⑥

WE ALSO CONSTRUCT THE TENSOR PRODUCT BUNDLE; THE FIBRE OF $\mathcal{E}\otimes\mathcal{F}$ IS $V_p \otimes W_p$, & THE TRANSITION FUNCTIONS ARE OF THE FORM

$$t_{ij}\otimes u_{ij}(p)\left(v_j \otimes w_j\right) \equiv \left(t_{ij}(p)\cdot v_j\right)\otimes\left(u_{ij}(p)\cdot w_j\right)$$

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A FIBRE BUNDLE WHOSE TYPICAL FIBRE IS ISOMORPHIC TO THE STRUCTURE GROUP IS CALLED A PRINCIPAL BUNDLE.

$$\begin{array}{ccc} \mathcal{G} & \hookrightarrow & P \\ & & \downarrow \\ & & M \end{array}$$

THE TRANSITION FUNCTIONS, $t_{ij}(p)\in\mathcal{G}$, ACT ON THE FIBRE FROM THE LEFT;

$$t_{ij}(p)\cdot g \qquad \text{for } g\in \mathcal{F}_p \simeq \mathcal{G}$$

BUT, WE CAN ALSO DEFINE AN ACTION OF $\mathcal{G}$ ON $P$ FROM THE RIGHT; CONSIDER $u\in P$, $\pi(u)=p$, & $p\in \mathcal{U}_i\cap\mathcal{U}_j$. THEN,

$$\begin{array}{ccc} & \overset{\phi_i}{\nearrow} & P \\ & & \downarrow \pi \\ \mathcal{U}_i\times\mathcal{G} & & M \end{array} \qquad \begin{aligned} u\cdot g &\equiv \phi_i^{-1}(u)\cdot g \\ &= (p, h_i)\cdot g \\ &\equiv (p, h_i g) \end{aligned}$$

NOTE THAT THIS IS COMPATIBLE WITH THE TRANSITION FUNCTIONS;

$$\begin{aligned} \phi_i^{-1}(u)\cdot g &= (p, h_i\cdot g) \\ &= (p, (t_{ij}h_j)g) \in \mathcal{U}_i\times\mathcal{G} \\ &= (p, h_j\cdot g)\in \mathcal{U}_j\times\mathcal{G} \end{aligned}$$

PRINCIPLE BUNDLES & VECTOR BUNDLES ARE CLOSELY RELATED; THEY MAY BE CONSTRUCTED from ONE ANOTHER.

LET $V$ BE A VECTOR SPACE, & $\rho : \mathcal{G} \to \mathrm{End}(V)$ A GROUP REP$^n$. THEN, if $P \to M$ IS A PRINCIPLE $\mathcal{G}$-BUNDLE, WE OBTAIN THE ASSOCIATED VECTOR BUNDLE

$$P \times_\rho V$$

By EXAMINING THE ACTION of $\mathcal{G}$ ON $P \times V$

$$g : (u, v) \mapsto (ug, \rho(g^{-1}) \cdot v)$$

THEN $P \times_\rho V = P \times V / \sim$, WHERE

$$(u, v) \sim (u', v') \quad \text{if } \exists\, g \text{ WITH}$$
$$u' = u \cdot g \ \& \ v' = \rho(g^{-1})\, v$$

THIS BEAUTIFUL CONSTRUCTION ACCOMPLISHES THE following; WHENEVER $\mathcal{U}_i \cap \mathcal{U}_j \neq \phi$ IN $M$, THE TRANSITION function for $P \times_\rho V$ IS

$$\rho(t_{ij})$$

THAT IS, WE HAVE A BUNDLE WHOSE fibres ARE $V$, & WHOSE TRANSITION functions ARE specified BY $P$.