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CATEGORY

~~ANOTHER EXAMPLE,~~

CATEGORIES WERE INVENTED TO formalize THE CONCEPT of A "NATURAL" CONSTRUCTION; ONE WHICH IS NOT DEPENDENT ON ARBITRARY CHOICES.

For example, for finite DIMENSIONAL VECTOR SPACES, $V \simeq V^*$. HOWEVER, TO ACTUALLY CONSTRUCT AN ISOMORPHISM ONE NEEDS TO fix A BASIS, I.E. MAKE AN ARBITRARY CHOICE.

CATEGORIES SOLVE THIS PROBLEM BY DEFININING "NATURAL"

$$\begin{array}{ccc} \mathrm{Id}(A) & \xrightarrow{\mathrm{Id}\,M} & \mathrm{Id}(B) \\ {\scriptstyle \tau(A)}\downarrow & & \downarrow{\scriptstyle \tau(A)} \\ A^{**} & \longrightarrow & B^{**} \end{array}$$

& τ IS JUST THE MAP $v \to e_v$.

THUS $(-)^{**}$ IS NATURAL; $\exists$ A NATURAL TRANSFORMATION of functors from Id TO IT.

Now WE COME TO A VERY powerful RESULT, YONEDA'S LEMMA. BASICALLY IT STATES THAT ANY OBJECT of ANY CATEGORY CAN BE RECOVERED from THE SET of ALL MAPS INTO THE OBJECT.

CONSIDER THE RIGHT-HOM functor

$$h_X \equiv \mathrm{Hom}_{\mathcal{C}}(-, X)$$

YONEDA'S LEMMA SAYS THAT <u>ALL</u> THE DATA of A manifold, A SCHEME, A <u>STACK</u>! IS CONTAINED IN h_X

MOST OBJECTS WHICH MATHEMATICIANS STUDY CONSIST of A SET TOGETHER WITH SOME SORT of ALGEBRAIC STRUCTURE ON TOP of IT; TOPOL^{l} SPACES, GROUPS, VECTOR SPACES, MANIFOLDS, BUNDLES,

EACH of THESE OBJECTS ~~HAVE~~ HAS THE NOTION of FUNCTION WHICH RESPECTS THE STRUCTURE; A CTS MAP, GROUP HOM, ~~VS HOM~~ LINEAR TRANS^{n}, ~~ETC~~ ETC.

A CATEGORY ENCOMPASSES BOTH THE "KIND of STRUCTURE" & "STRUCTURE-PRESERVING FUNCTIONS".

A CATEGORY $\mathcal{C}$ CONSISTS of

(1) A COLLECTION of OBJECTS $ob(\mathcal{C})$

(2) $\forall$ ORDERED PAIRS (A,B) of OBJECTS of $\mathcal{C}$, $Hom(A,B)$ A <u>SET</u> of MORPHISMS from $A \to B$ (CALLED ARROWS)

(3) A FUNCTION $\circ : Hom(A,B) \times Hom(B,C) \to Hom(A,C)$

SATISFYING THE FOLLOWING AXIOMS; if $A,B,C,D \in ob(\mathcal{C})$

(A1) $Hom(A,B) \cap Hom(C,D) = \phi$ if $A \neq C$ OR $B \neq D$

(A2) (ASSOCIATIVITY) for $f \in Hom(A,B)$, $g \in Hom(B,C)$ & $h \in Hom(C,D)$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

(A3) (IDENTITY) $\forall A \in ob(\mathcal{C})$, $\exists\ Id_A \in Hom(A,A)$, SATISFYING $f \circ Id_A = f\ \forall f \in Hom(A,B)$ & $Id_A \circ g = g\ \forall g \in Hom(B,A)$

LETS LOOK AT SOME EXAMPLES OF A CATEGORY.

((Top)) THE CATEGORY OF ALL TOPOLOGICAL SPACES, WITH MORPHISMS BEING CONTINUOUS FUNCTIONS

((Ab)) THE CATEGORY OF ABELIAN GRPS; MORPHISMS = GRP HOMS.

((Top_X)) THE TOPOLOGICAL CATEGORY ON $(X, \mathcal{T})$

$\mathcal{O}b$ = OPEN SUBSETS OF X (ELEMENTS OF $\mathcal{T}$)

$\mathcal{M}or$ = INCLUSIONS.

NOTE THAT IF $U \neq V$, $\mathrm{Hom}(U,V) = \mathrm{Hom}(V,U) = \emptyset$,

& WE ALWAYS HAVE $i: U \hookrightarrow U = id_U \in \mathrm{Hom}(U,U)$

BEFORE I TALK ABOUT WHAT A MAP BETWEEN CATEGORIES IS, LET'S REVIEW THE NOTION OF A COMMUTATIVE DIAGRAM.

LET A, B, C, D BE SETS & $\alpha, \beta, \gamma, \delta$ BE CTS FUNCTS

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\delta"] \\ C \arrow[r, "\beta"'] & D \end{tikzcd}$$

THEN THE DIAGRAM IS COMMUTATIVE, OR CARTESIAN, IF

$$\delta(\alpha(a)) = \beta(\gamma(a)) \quad \forall\ a \in A.$$

IN A GENERAL CATEGORY, $ABCD$ ARE OBJECTS, $\alpha\beta\gamma\delta$ ARE MORPHISMS, & A COMMUTATIVE DIAGRAM IS THE STATEMENT

$$\delta \circ \alpha = \beta \circ \gamma \in \mathrm{Hom}(A, D)$$

A map between Categories is called a functor. ~~functors come~~ Remember that categories consist of objects & morphisms between them; thus a functor not only acts on obj, but morphisms as well. Functors come in two varieties; covariant & contravariant.

Covariant functor $T : \mathcal{C} \to \mathcal{D}$ is an assignment $X \mapsto T(X)$ for each object $X \in ob\,\mathcal{C}$ to an obj $T(X) \in ob\,\mathcal{D}$ & an assignment $\forall\, f \in Hom_{\mathcal{C}}(A, B)$
$$f \mapsto T(f) \in Hom_{\mathcal{D}}(T(A), T(B)), \text{ such that}$$

(1) $T(id_A) = id_{T(A)}$

(2) $T(g \circ f) = T(g) \circ T(f)$

A covariant functor $T' : \mathcal{C} \to \mathcal{D}$ is a functor with "reversed arrows": if, $f \in Hom_{\mathcal{C}}(A,B)$, then $T'(f) \in Hom_{\mathcal{D}}(T'(B), T'(A))$;

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \Big\downarrow T & \\ T(A) & \xleftarrow{T(f)} & T(B) \end{array}$$

& composition is reversed; $T(f \circ g) = T(g) \circ T(f)$

Lets give some examples

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TWO <u>VERY</u> IMPORTANT functors ARE THE Hom functors. fix A CATEGORY $\mathcal{C}$ & AN OBJECT $X \in Ob(\mathcal{C})$. THEN $\exists$ TWO NATURAL functors

$$\mathcal{C} \rightsquigarrow ((SETS))$$

$$Hom_{\mathcal{C}}(X, -) \quad \& \quad Hom_{\mathcal{C}}(-, X).$$

CLEARLY, for EACH OBJECT $Y \in Ob(\mathcal{C})$, WE GET A SET. LET $f \in Hom_{\mathcal{C}}(Y, Z)$. WE WANT AN ELEMENT of

$$Hom_{((SETS))}\big(Hom_{\mathcal{C}}(X, Y),\ Hom_{\mathcal{C}}(X, Z)\big).$$

THAT IS, GIVEN $f : Y \to Z$, WE WANT

$$\begin{array}{ccc} X & & X \\ \downarrow & \rightsquigarrow & \downarrow \\ Y & & Z \end{array}$$

CLEARLY THE THING TO DO IS composition! GIVEN $g \in Hom_{\mathcal{C}}(X, Y)$, WE compose $f \circ g$ TO GET AN ELEMENT of $Hom_{\mathcal{C}}(X, Z)$. THUS THE $Hom(X, -)$ functor SENDS $f \rightsquigarrow f \circ$

WHAT ABOUT $Hom_{\mathcal{C}}(-, X)$? GIVEN $f : Y \to Z$ & ELEMS of $Hom(Y, X)$ & $Hom(Y, Z)$;

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & & X \end{array}$$

GIVEN h, CAN GET g?
NO; GIVEN g CAN GET h

THE THING TO DO IS CLEAR; GIVEN $g \in Hom(Z, X)$ WE

HAVE AN ELEM of $Hom(Y, X)$; $f \circ g$.

THUS, $Hom_C(Y, Z) \rightsquigarrow Hom_{((SETS))}\Big(Hom_C(Z, X),\ Hom_C(Y, X)\Big)$

I.E. $Hom(-, X)$ IS CONTRAVARIANT, WHILE
$Hom(X, -)$ IS COVARIANT.

LETS LOOK AT ANOTHER EXAMPLE of A ~~COVA~~ CONTRAVARIANT functor; SHEAVES

A SHEAF of ABELIAN GROUPS ON A SPACE X IS A functor from $((Top_X)) \rightsquigarrow ((Ab))$ & SATISFYING A CONDITION;

$U \in Ob((Top_X))$ $U \mapsto \mathcal{F}(U)$ AN AB GRP.

if $V \overset{i}{\hookrightarrow} U$ THEN $\exists$ A MORPHISM $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$

I.E. THE SHEAF functor IS CONTRAVARIANT.

TO IMPOSE THE LAST TWO parts of the Definitions of A SHEAF, WE MUST REQUIRE THAT $\mathcal{F}(\bigcup_i U_i)$ IS A CERTAIN LIMIT; THE EXACT form IS NOT IMPORTANT, JUST THAT IT CAN BE STATED purely IN CATEGORY-THEORETIC TERMS.

ANOTHER IMPORTANT functor IS THE GLOBAL SECTION functor. CONSIDER THE CATEGORY of SHEAVES ON X; OBJECTS ARE SHEAVES of ABELIAN GROUPS, MORPHISMS ARE SHEAF MORPHISMS. THEN Define THE functor $F : ((Sh_X)) \rightsquigarrow ((Ab))$ BY

$$F(\mathcal{F}) = \Gamma(X, \mathcal{F})$$

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Consider something we use all the time; a similarity transformation, eg from gauge theory.

$$T = \tau^{-1} S \tau$$

We would like to know; what is the most general way to make a gauge transformation? Lets also consider the case where τ^{-1} may not make sense;

$$\tau T = S \tau$$

The really interesting viewpoint is to consider T & S as some sort of map. Then T & S should be "images" of the underlying equivalence class f. Take f to be

$$c \xrightarrow{f} c'$$

As written, T & S need not be maps between the same spaces even! ~~For example, consider a group action~~ on ~~the sections of a bundle, which is "compatible"~~ with ~~the action on the base; take $\mathcal{O}_{\mathbb{P}^1}(1)$ & $G = \mathbb{C}^*$. Compatibility implies that~~

$$\cancel{g \cdot \pi(p) = \pi \cdot g(p)}$$

for this to be possible, there should be a notion of $\tau(c)$ & $\tau(c')$

Consider a grp action on a bundle compatible w/ the action on the base; for $p \in \mathcal{E} \to B$,

$$g \cdot \pi(p) = \pi(g \cdot p)$$

THUS, WE HAVE $\cancel{S(c) \xrightarrow{T(c)} S(c')}$

$$S(c) \xrightarrow{\tau(c)} T(c)$$

$$S(c') \xrightarrow{\tau(c')} T(c')$$

I.E. A SIMILARITY TRANSFORMATION IS ONE IN WHICH $\exists$ TWO MAPS $S(f)$ & $T(f)$, WHICH ARE COMPOSED AS

$$S(c) \xrightarrow{S(f)} S(c') \xrightarrow{\tau(c')} T(c')$$

OR EQUALLY

$$S(c) \xrightarrow{\tau(c)} T(c) \xrightarrow{T(f)} T(c')$$

YOU MAY HAVE GUESSED THAT WHEN I DRESS THIS UP IN FANCY LANGUAGE, S & T ARE FUNCTORS. τ IS CALLED A NATURAL TRANSFORMATION OF FUNCTORS:

GIVEN A CATEGORY $\mathcal{C}$ & $\mathcal{D}$ & FUNCTORS $S, T : \mathcal{C} \rightsquigarrow \mathcal{D}$, THEN A NATURAL TRANSFORMATION $\tau : S \to T$ IS AN ASSIGNMENT $\forall\ A \in Ob\,\mathcal{C}$ TO AN ELEMENT OF $\mathrm{Hom}_{\mathcal{D}}(S(A), T(A))$ SUCH THAT $\forall\ f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$,

$$\begin{array}{ccc} S(A) & \xrightarrow{S(f)} & S(B) \\ \tau(A)\downarrow & & \downarrow\tau(B) \\ T(A) & \xrightarrow{T(f)} & T(B) \end{array}$$

COMMUTES

CONSIDER MY EARIER COMMENT ABOUT VECTOR SPACES; IN GENERAL (EVEN for $\dim = \infty$), THERE IS A MAP BETWEEN V & V^{**};

$$V^* \equiv \mathrm{Hom}_k(V, k)$$ IN THE CATEGORY of k-VECTOR SPACES.

$$V^{**} \equiv \mathrm{Hom}_k(\mathrm{Hom}_k(V, k), k)$$

& for $v \in V$ Define $e_v : V^* \to k$ By $e_v(\lambda) = \lambda(v)$

INDEED, $(-)^{**} : \mathrm{Vect}_k \rightsquigarrow \mathrm{Vect}_k$ IS A functor!

$$\begin{tikzcd} V \arrow[r, "M"] \arrow[d, "\wr"] & W \arrow[d] \\ V^{**} \arrow[r, "M^{**}"] & W^{**} \end{tikzcd}$$

$$v \to Mv$$
$$e_v \xrightarrow{M^{**}} e_{Mv} \equiv \quad e_{Mv}(\lambda) = \lambda(Mv)$$
for $\lambda \in W^{**}$

CLEARLY ITS k-LINEAR.

Now CONSIDER THE IDENTITY functor, $\mathrm{Id}_{\mathcal{C}}$

$\mathrm{Id}_{\mathcal{C}} : A \rightsquigarrow A$ for $A \in \mathrm{Ob}\,\mathcal{C}$

$\mathrm{Id}_{\mathcal{C}} : f \rightsquigarrow f$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$