①

Flows & Forms

Last time, we saw how a curve in a manifold has an associated vector field, the velocity. Specifically, for $c: \mathbb{R} \to M$ passing though a coördinate neighborhood U,

$$X^\mu_{x(t)} = \frac{dx^\mu(t)}{dt} \equiv d_t \, x^\mu \circ \varphi_U \circ c(t)$$

Consider this, on $\varphi_U(U) \subset \mathbb{R}^m$, we have a collection of m ODE's,

$$d_t x^\mu = X^\mu(t) \quad \left(\text{or} \quad d_t f(t) = X(f(t)) \right)$$

So, there is a unique solution locally on $\varphi_U(U)$, by the existence & uniqueness of ODEs, once we specify an inital condition. Lets write these solutions as $\sigma^\mu(t)$, with $\sigma^\mu(t=t_0) \equiv x^\mu_0$

$$\frac{d\sigma^\mu(t)}{dt} = X^\mu(\sigma(t))$$

Since the X^μ transform in the same way as the x^μ, we see that the solutions in different coördinate systems match up in the expected way, so that we in fact have a map

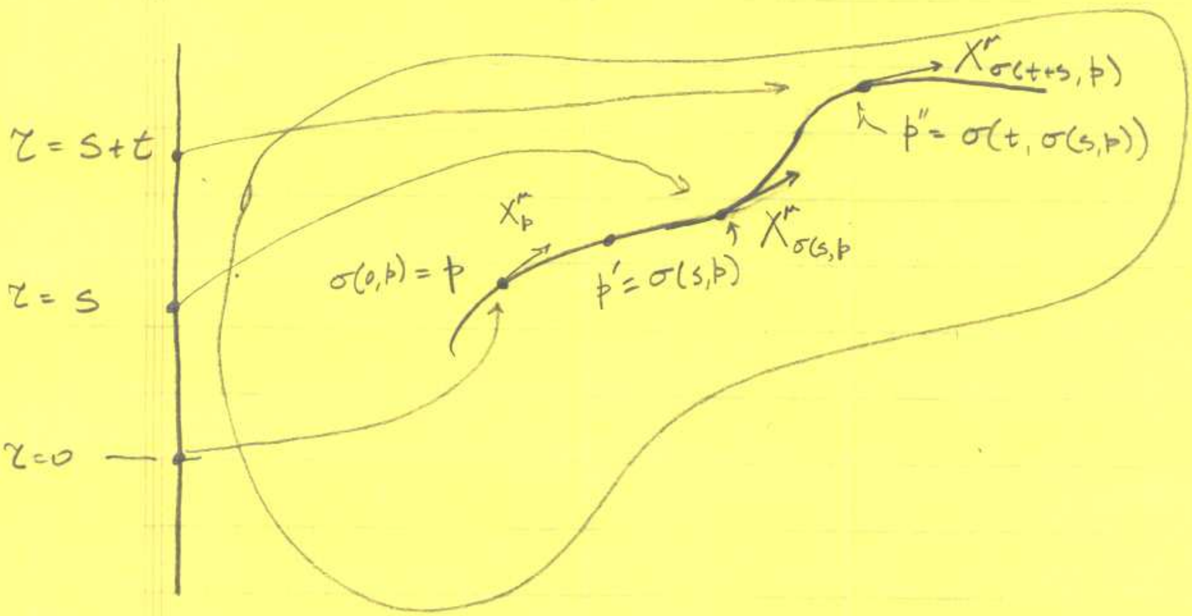
$$\sigma : \mathbb{R} \times M \to M$$

Time   initial conditions

called the flow generated by the vector field $\underline{X} \in \mathfrak{X}(M)$

Note that flows satisfy a composition law; beginning at a point p, if we flow to a new point $\sigma(s,p)$ & flow again for a time t,

$$\sigma(t, \sigma(s,p)) = \sigma(t+s, p)$$



This is clear in local coördinates; if $p'' \in U$ & $\varphi_u : U \to \mathbb{R}^m$, write $\sigma^\mu = x^\mu \circ \varphi_u \circ \sigma$ for local coördinates x^μ,

$$\frac{d}{dt}\sigma^\mu(t, p') = X^\mu(\sigma(t, p'))$$

$$\frac{d}{dt}\sigma^\mu(s+t, p) = X^\mu(\sigma(t+s, p))$$

& $\sigma^\mu(0, p') = p'$

$\sigma^\mu(0+s, p) = p$

Thus, the two expressions satisfy the same ODE & initial conditions, so they must be identical.

LETS LOOK AT AN EXAMPLE ON $\mathbb{R}^2$, WITH THE VECTOR FIELD $x\partial_y + y\partial_x = \underline{X}$

Now, $X^y = x$ & $X^x = y$, SO WE MUST HAVE FUNCTIONS σ^x & σ^y SATISFYING

$$d_t \sigma^i = X^i(\sigma^x, \sigma^y)$$

OR

$$d_t \sigma^x = X^x(\sigma) = \sigma^y$$
$$d_t \sigma^y = \sigma^x$$

WHICH GIVE $d_t^2 \sigma^x = \sigma^x$ & $d_t^2 \sigma^y = \sigma^y$. THESE HAVE SOLUTIONS

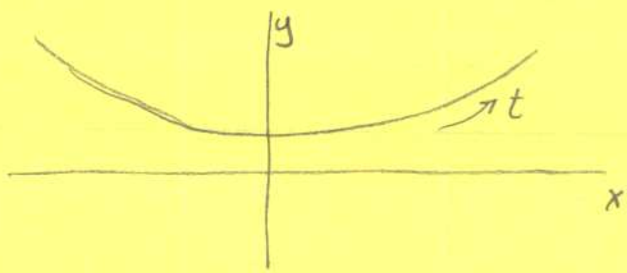
$$\sigma^x = Ae^t + Be^{-t}$$
$$\sigma^y = Ae^t - Be^{-t}$$

WHERE A & B ARE FIXED BY INITIAL CONDITIONS. IF

$\sigma^x(0) = 0$, $A = -B$

$\sigma^y(0) = 1$, $A = -B = \frac{1}{2}$ $\Rightarrow$ $\sigma = (\sinh t, \cosh t)$

$\Rightarrow$



THE PREVIOUSLY MENTIONED COMPOSITION PROPERTY TELLS US THAT THE FLOW AT A GIVEN TIME DEFINES AN ELEMENT OF A GROUP. IF $\underline{X} \in \mathfrak{X}(M)$ & σ IS THE CORRESPONDING FLOW, THEN FOR ANY TIME t,

$$\sigma(t) : M \to M.$$

THINKING OF THE FLOWS AS A SET OF MAPS $M \to M$ IN 1-1 CORRESPONDENCE WITH $\mathbb{R}$, WE SEE THAT COMPOSITION OF FLOWS CORRESPONDS TO ADDITION IN $\mathbb{R}$;

(i) $\sigma_t \circ \sigma_s(p) = \sigma_{t+s}(p)$

(ii) $\sigma_0(p) = p$

(iii) $\sigma_{-t}(p) = \sigma_t^{-1}(p)$

OR, THERE IS A GROUP HOMOMORPHISM

$$\mathbb{R} \longmapsto \{\sigma : M \to M\}$$

THIS GROUP IS CALLED A ONE-PARAMETER GROUP OF TRANSFORMATIONS.

LETS THINK ABOUT THE FLOW LOCALLY NOW; $\sigma_t^\mu(x) = x^\mu \circ \varphi_u \circ \sigma(t, x)$ THEN, IF WE LOOK AT A SMALL TIME ϵ, WE CAN TAYLOR EXPAND THESE FUNCTIONS

$$\sigma_\epsilon^\mu(x) = x^\mu + \epsilon\, d_t \sigma_{x_0}^\mu(0) + \mathcal{O}(\epsilon^2)$$
$$= x_0^\mu + \epsilon X^\mu$$

⑤

THEN, WE CAN THINK OF $\sigma^\mu(\epsilon)$ AS A LOCAL TRANSFORMATION OF THE COÖRDINATES x^μ, & THE TRANSFORMATION IS GENERATED BY THE VECTOR X^μ.

CONTINUING THE EXPANSION, WE FIND

$$\sigma_t^\mu(x) = x_0^\mu + t\,\frac{d\sigma^\mu}{dt}\Big|_{t=0} + \frac{1}{2}t^2\frac{d^2\sigma^\mu}{dt^2}\Big|_{t=0} + \cdots$$

$$= x_0^\mu + t\,X^\mu(\sigma_0(x)) + \frac{t^2}{2}\frac{d}{dt}X^\mu(\sigma_t(x))\Big|_{t=0} + \cdots$$

$$= x_0^\mu + tX^\mu + \frac{t^2}{2}\left(\frac{d\sigma^\nu}{dt}\partial_\nu X^\mu\right)\Big|_{t=0} + \cdots$$

$$= x_0^\mu + tX^\mu + \frac{t^2}{2}X^\nu\partial_\nu X^\mu + \cdots$$

WHICH LOOKS LIKE $1 + t\underline{X}[x_0] + \frac{t^2}{2}\underline{X}\left[\underline{X}[x_0^\mu]\right] + \cdots$
WHICH WE DEFINE TO BE

$$\sigma^\mu = e^{t\underline{X}}[x_0^\mu] = e^{t\underline{X}}[x_0^\mu]$$

THUS, THE FLOW $\sigma^\mu(t,x)$ IS THE EXPONENTIATION OF $\underline{X}$ ACTING ON THE INITIAL POINT'S COÖRDINATE FUNCTION.

We've explored the way curves & vector fields are related, so vector fields should be somewhat familiar now. Last week, we established that the set of all vector fields is a vector space. We will now discuss $\mathfrak{X}(M)$'s dual space.

First, what is a dual space? It is the vector space of linear <u>functions</u> $V \longmapsto \mathbb{R}$ (or $\mathbb{C}$). That is, if $f \in V^{\vee}$, the dual space to V,

$$f(v+w) = f(v) + f(w)$$
$$f(r \cdot v) = r \cdot f(v)$$

The dual space to the space of vector fields has a special name, the space of one-forms $\Omega(M)$. These forms are written as df, & their action on vector fields is defined to be

$$\langle df, X \rangle \equiv X[f]$$
$$\sim X^{\mu} \partial_{\mu} f$$

on the basis vectors for $\mathfrak{X}(M)$: we have

$$\langle df, \partial_{x^{\mu}} \rangle = \frac{\partial f}{\partial x^{\mu}}$$

So we can write a dual basis to $\{\partial_{x^{\mu}}\}$ as dx^{μ}

$$\langle dx^{\nu}, \partial_{x^{\mu}} \rangle = \delta^{\nu}{}_{\mu}$$

THEN, AN ARBITRARY form $\omega \in \Omega(M)$ TAKES THE LOCAL form $\omega = \omega_\mu dx^\mu$ for SOME functions ω_μ, AND IT'S ACTION ON A VECTOR FIELD IS LOCALLY EXPRESSED AS

$$\begin{aligned} \langle \omega_\mu dx^\mu, X^\nu \partial_\nu \rangle &\overset{\text{LINEARITY}}{=} \omega_\mu X^\nu \langle dx^\mu, \partial_{x^\nu} \rangle \\ &= \omega_\mu X^\nu \delta^\mu{}_\nu \\ &= \omega_\mu X^\mu \end{aligned}$$

So far, WE'VE DISCUSSED VECTORS IN TERMS of THE SPACE of VECTOR FIELDS $\mathfrak{X}(M)$. AN ALTERNATIVE VIEWPOINT, WHICH WILL SEGUE INTO BUNDLES NEXT WEEK, IS TO CONSIDER THE VECTOR AT A POINT DEFINED AS LAST WEEK'S PRESENTATION IN TERMS of AN EQUIVALENCE CLASS of CURVES. THIS ASSOCIATES TO EVERY POINT $p \in M$ A VECTOR SPACE T_pM, THE TANGENT SPACE AT p. If x^μ ARE SOME LOCAL COÖRDINATES AT p, T_pM IS THE SPAN of $\{\partial_{x^\mu}\}$. THE DUAL SPACE IS WRITTEN T_p^*M, & IS SPANNED BY $\{dx^\mu\}$.

LAST SEMESTER, WE DISCUSSED MODULES, VECTOR SPACES, AND THEIR TENSOR PRODUCTS. FOR EACH POINT p, WE CAN CONSTRUCT A TENSOR PRODUCT of r copies of T_pM & s copies of T_p^*M:

$$\underbrace{T_pM \otimes \cdots \otimes T_pM}_{r} \otimes \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{s}$$

AN (r,s) - TENSOR IS A MULTILINEAR MAP

$$T_p : T_pM \otimes \cdots \otimes T_pM \otimes T_p^*M \otimes \cdots \otimes T_p^*M \longrightarrow \mathbb{R}$$

SATISFYING

(i) $T_p(\cdots, \omega + \sigma, \cdots) = T(\cdots, \omega, \cdots) + T(\cdots, \sigma, \cdots)$
(ii) $T_p(\cdots, \underline{X} + \underline{Y}, \cdots) = T(\cdots, \underline{X}, \cdots) + T(\cdots, \underline{Y}, \cdots)$
(iii) $T_p(\cdots, a\omega, \cdots, b\underline{X}, \cdots) = ab\, T(\cdots, \omega, \cdots, \underline{X}, \cdots)$

$\forall \quad \omega, \sigma \in T_p^*M, \quad \underline{X}, \underline{Y} \in T_pM, \quad a, b \in \mathbb{R}$

WE DEFINE THE COMPONENTS OF A TENSOR TO BE ITS ACTION ON THE BASIS VECTORS IN A COÖRDINATE NEIGHBORHOOD;

$$T_p{}^{\mu_1 \cdots \mu_s}{}_{\mu_{s+1} \cdots \mu_{s+r}} = T_p\left(dx^{\mu_1}, \cdots dx^{\mu_s}, \frac{\partial}{\partial x^{\mu_{s+1}}}, \cdots, \frac{\partial}{\partial x^{\mu_{r+s}}}\right)$$

THUS, IN LOCAL COÖRDINATES, WE CAN WRITE

$$T(\omega^1, \omega^2, \cdots, \omega^s, X_{s+1}, \cdots, X_{r+s}) = \left(\omega^1_{\mu_1} \omega^2_{\mu_2} \cdots \omega^s_{\mu_s} X^{\mu_{1+s}}_{1+s} \cdots X^{\mu_{s+r}}_{s+r}\right) \times T^{\mu_1 \cdots \mu_s}{}_{\mu_{s+1} \cdots \mu_{r+s}},$$

BY PROPERTY (iii). WE CAN ALSO DIFFERENTIATE TENSORS WITH EXPLICIT BEHAVIOR ON THE INTERCHANGE OF TWO ARGUMENTS; IF

$$T(\cdots, \overset{i}{X}, \cdots, \overset{j}{Y}, \cdots) = T(\cdots, \overset{i}{Y}, \cdots, \overset{j}{X}, \cdots)$$

FOR ALL $X, Y \in \mathfrak{X}(M)$, T IS SAID TO BE SYMMETRIC IN POSITIONS i & j. IF T IS SYMMETRIC IN ALL POSITIONS, T is said to be fully symmetric, or more simply symmetric. Likewise if T acquires a negative sign, it is said to be anti-symmetric.

A totally anti-symmetric $(0,r)$-tensor is called a differential form of order r, or an r-form. It is written in terms of components as a completely anti-symmetric sum of tensor products of basis covectors;

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \equiv \sum (-1)^P \, dx^{\mu_{P(1)}} \otimes dx^{\mu_{P(2)}} \otimes \cdots \otimes dx^{\mu_{P(r)}}$$

r-forms satisfy the axioms of a vector space, & are denoted by $\Omega_p^r(M)$. An element ω of $\Omega_p^r(M)$ is written in terms of basis elements as above:

$$\omega \equiv \frac{1}{r!} \omega_{\mu_1 \cdots \mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$$