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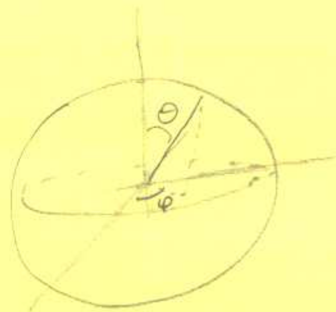
Manifolds

Today we'll discuss manifolds, & some structures on them. I'll tell you what physicists actually mean when we write things like

$$\phi(x^\mu)$$

for functions or vectors (→ next week!) on some manifold. First, lets give a simple motivation, S^2. We all know that given any point on S^2, angles θ & φ may be given to describe the point.



The problem is, the specification is not unique, $(\theta, \varphi) = (0, a), (0, b)$ & $(\theta, \varphi) = (3, 0), (3, 2\pi)$ define the same points with different numbers. We cannot describe coördinates in the usual way, or do differiation as expected; what happens to $\partial_\varphi f$ as $\theta \to 0$? To be well defined, it must go to zero, but a coördinate shouldn't depend on other coördinates.

The definition of a manifold avoids these problems by introducing "local coördinates"

A manifold M is

(1) A topological space $(M, \mathcal{T})$

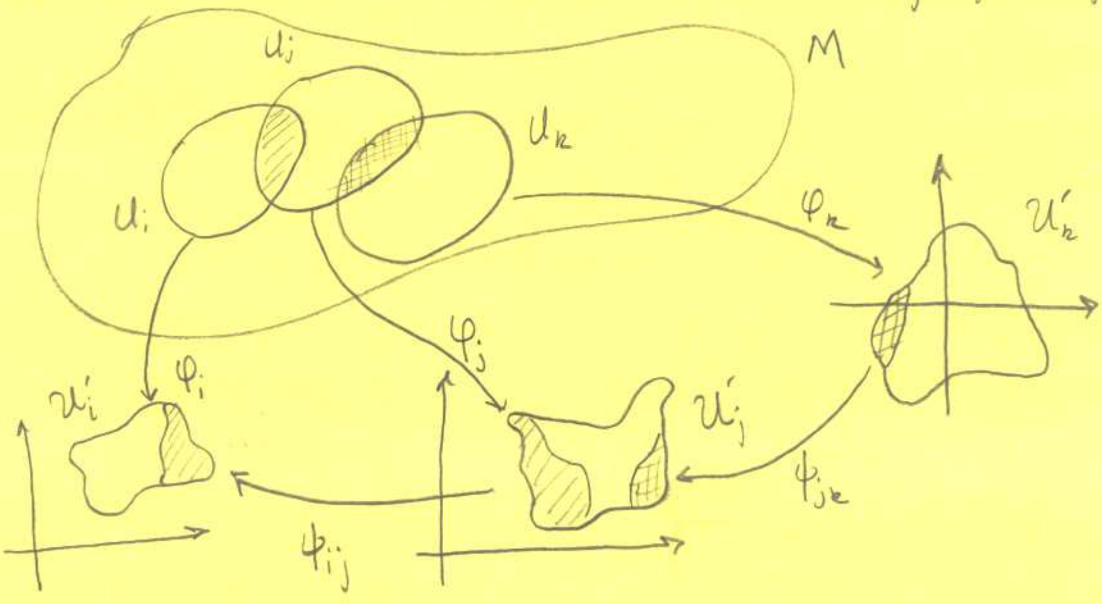
(2) A family of open sets $\{U_i\} \subseteq \mathcal{T}$, with $\bigcup_i U_i = M$ & homeomorphisms $\{\varphi_i\}$ (cts fcts w/ cts inverses)

$$\varphi_i : U_i \to \mathcal{U}'_i \subset \mathbb{R}^n$$

to open sets $\mathcal{U}'_i$ in $\mathbb{R}^n$. Basically this says that φ_i "elastically" deforms U_i into $\mathcal{U}'_i$.

(3) For each $U_{ij} = U_i \cap U_j$, $U_{ij} \neq \phi$, there exists a smooth (∞^{ly} differentiably) map

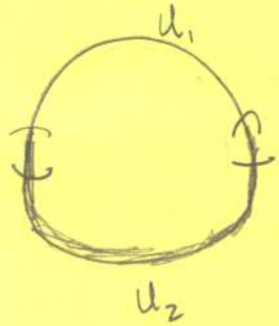
$$\psi_{ij} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$



Note; Because φ_i & φ_i^{-1} are cts, $\varphi_i(U_i \cap U_j) \subseteq \mathcal{U}'_i$ is open! Furthermore ψ_{ij} is only defined for the marked area inside $\mathcal{U}'_j$ ... it need not make sense elsewhere!

③

LETS DO THE SIMPLEST NON-TRIVIAL EXAMPLE, S^1.



$$\mathcal{U}_1 = (-\epsilon, \pi+\epsilon)$$
$$\mathcal{U}_2 = (\pi-\epsilon, 2\pi+\epsilon)$$

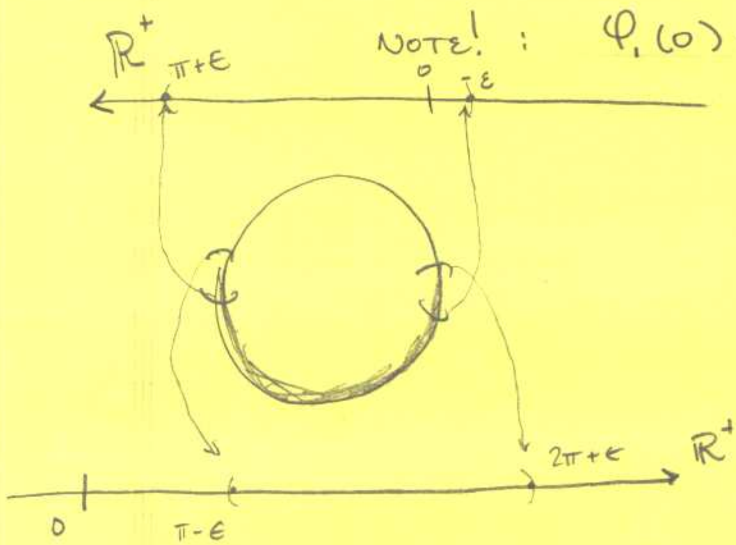
$$\text{\&} \quad \mathcal{U}_1 \cap \mathcal{U}_2 = (-\epsilon, \epsilon) \cup (\pi-\epsilon, \pi+\epsilon)$$

ONE CAN CHECK THAT $\left(S^1, \{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_1 \cap \mathcal{U}_2\}\right)$ IS A TOPOLOGICAL SPACE. NOW, WE CHOOSE OUR OPEN SETS TO BE $\{\mathcal{U}_1, \mathcal{U}_2\}$, & WE MUST DEFINE HOMEOMORPHISMS

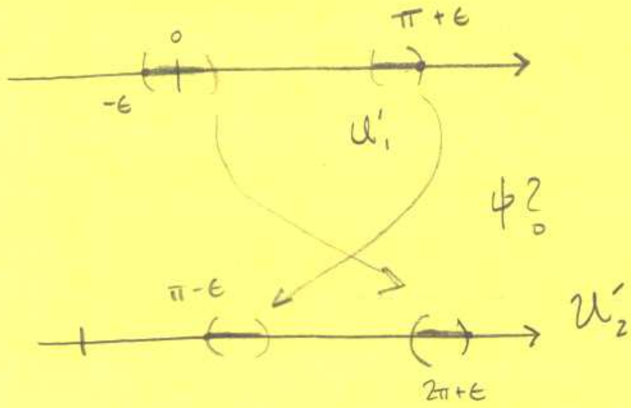
$$\varphi_1 : \mathcal{U}_1 \to \mathbb{R}$$
$$\varphi_2 : \mathcal{U}_2 \to \mathbb{R};$$

THIS IS EASY, SET $\varphi_1(\theta) = \theta$ & $\varphi_2(\theta) = \theta$

NOTE! : $\varphi_1(0) = 0$, $\varphi_2(0) = 2\pi$!!



So, we have



$\xi\ \psi_{21}$ has to reverse the order!

$\psi_{21}(\pi+\epsilon) = \pi+\epsilon$

$\psi_{21}(\pi-\epsilon) = \pi-\epsilon$

$\psi_{21}(-\epsilon) = 2\pi-\epsilon$

$\psi_{21}(\epsilon) = 2\pi+\epsilon$

$c\,\mathcal{U}'_1$

Remember, on $[\epsilon, \pi-\epsilon]$ ψ_{21} does not even have to make sense!

$$\psi_{21}(x) = (2\pi + x)\,\Theta(\pi/2 - x) + x\,\Theta(x - \pi/2)$$

$$\psi_{21}(x) = \left\{ \begin{array}{c|c} 2\pi + x & x < \epsilon \\ x & x > \pi - \epsilon \end{array} \right\}$$

etc.

Note, $$\partial_x \psi_{21} = \Theta(\pi/2 - x) + \Theta(x - \pi/2) + (2\pi + x)\,\delta(\pi/2 - x) + x\,\delta(x - \pi/2)$$

Apparently it's not smooth! But thats ok, b/c it's derivatives are well defined on $(-\epsilon, \epsilon) \cup (\pi - \epsilon, \pi + \epsilon)$

REMEMBER THAT M WAS A TOPOLOGICAL SPACE; JUST A SET OF POINTS & SOME SUBSETS.

If $p \in U_i$, U_i IS CALLED A COÖRDINATE NEIGHBOURHOOD of p, & $\{U_i, \varphi_i\}$ IS CALLED A CHART.

If WE WANT TO TALK ABOUT THE <u>COÖRDINATES</u> of p, WE NEED TO THINK A BIT MORE ABOUT WHAT A COÖRDINATE IS. IN $\mathbb{R}^n$, WE HAVE COÖRDINATE FUNCTIONS,

$$x^i : \mathbb{R}^n \longrightarrow \mathbb{R}$$

WHICH TAKE A POINT

$$x^i : \underbrace{(\cdots, 0, 1, \overset{i^{th}}{15}, -10, 3, \cdots)}_{n} \longmapsto 15 .$$

THAT IS, THEY TAKE A POINT IN $\mathbb{R}^n$ & GIVE ITS DISTANCE ALONG THE i^{th} AXIS.

THEN, for $p \in U_i \subset M$, WE HAVE THAT ITS COÖRDINATES IN U_i ARE $\{x^j \circ \varphi_i\}$;

$$(x^1 \circ \varphi_i , x^2 \circ \varphi_i , x^3 \circ \varphi_i , \cdots , x^n \circ \varphi_i)$$

THIS WAY of THINKING ABOUT COÖRDINATES, AS LOCALLY DEFINED FUNCTIONS, IS INTRINSIC TO A SCHEME-THEORETIC VIEWPOINT, WHICH I WONT COVER, if AT ALL, UNTIL MUCH LATER.

In string theory, bosons are thought of as coördinates on the target manifold X. Intrinsically, however, they are maps

$$\phi : \Sigma \to X.$$

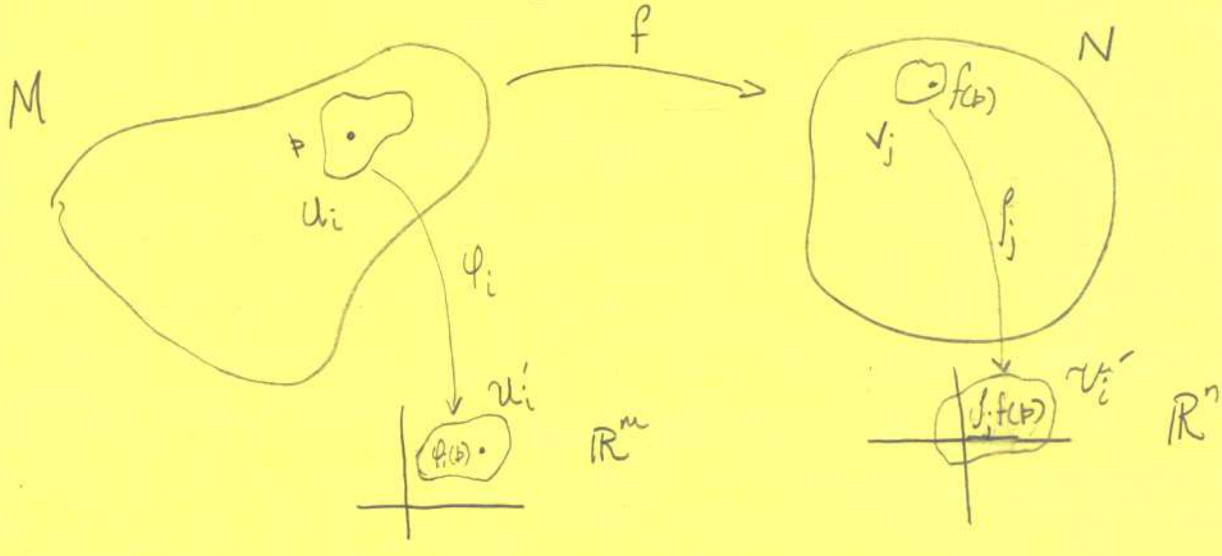
How then, do we arrive at the familiar form

$$\partial_z \phi^\mu(z,\bar z)\, \partial_{\bar z} \phi^\nu(z,\bar z) \,?$$

Lets consider the general case, M & N are manifolds of respective dimension m & n, &

$$f : M \to N.$$

Then for each $p \in M$, we have $f(p) \in N$



If U_i & V_j are respectively coördinate neighbourhoods of p & $f(p)$, then we obtain points

$$\varphi_i(p) \in U_i' \subset \mathbb{R}^m \quad \& \quad \rho_j(f(p)) \in V_j' \subset \mathbb{R}^n$$

By specifying coördinate functions

$$(x^1, x^2, \cdots x^m) \text{ \& } (y^1, y^2, \cdots y^n)$$

The points become $x^\mu \circ \varphi_i(p)$ & $y^\alpha \circ \rho_j(f(p))$, which are commonly written as

$$\varphi^\mu(p) \text{ \& } \rho_j^\alpha(f(p)).$$

When physicists are working in a single coördinate chart, we would just write φ^μ & ρ^α.

Now, let's rewrite p in a complicated way:

$$p = \varphi_i^{-1} \circ \varphi_i(p)$$
$$\text{"="}\ \varphi_i^{-1} \circ x^\mu(p)$$

I write "=" here because there are a few extra steps I'm hiding, but they are messy & the meaning should be clear. A good exercise is to work out exactly what this means.

Now, we can write f "in coordinates" as a map from $\mathcal{U}_i'$ to $\mathcal{V}_j'$

$$(y^\alpha \circ \rho_j) \circ f \circ \varphi_i^{-1} \circ x^\mu(p)$$

or as physicists would write it, $f^\alpha(x^\mu)$

NOW, WE ALSO WANT TO SAY WHEN A MAP $f: M \to N$ IS DIFFERENTIABLE:

If $\rho_j \circ f \circ \varphi_i^{-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ IS AN ∞-DIFF$^{\text{BLE}}$ $\text{f}_{\text{CT}}{}^{\text{n}}$ IN THE TRADITIONAL SENSE ($\partial_\mu^r f^\alpha(x)$ IS DEF$^{\text{d}}$ $\forall\, r \geq 0$) for ANY COÖRDINATE CHARTS $\mathcal{U}_i$ & V_j, f IS SMOOTH.

THIS IS WHERE THE MAGIC of THE DEFINITION of MANIFOLDS SHINES; SINCE ψ_{ij} WAS ∞^{ly} DIFF$^{\text{le}}$, if f IS SMOOTH IN JUST ONE CHART, IT IS SMOOTH IN ALL CHARTS.

WE NOW HAVE A RIGOROUS DEF$^{\text{n}}$ FOR THE PHYSICS EXPRESSION

$$\varphi^\mu(z, \bar{z})$$

(z & $\bar{z}$ ARE (COMPLEX) COÖRDINATE FUNCTIONS). TO UNDERSTAND $\partial_z \varphi^\mu$, IT TURNS OUT, WILL REQUIRE MORE MACHINERY: WE MUST UNDERSTAND VECTOR FIELDS.

IN PHYSICS, WE ARE TAUGHT THAT VECTORS ARE OBJECTS WITH INDICIES WHICH TRANSFORM IN A CERTAIN WAY UNDER COÖRDINATE TRANSFORMATIONS. AS I WAS TOLD,

"VECTORS ARE THINGS THAT TRANSFORM LIKE VECTORS"

We will take, at first, a more abstract approach to vectors, & derive the more familiar form.

Definition; A smooth function on a manifold M is a map $f : M \to \mathbb{R}$, that is, a smooth assignment of