①

RECALL THAT A PRESHEAF IS A COLLECTION OF ASSIGNMENTS

$$U \mapsto \mathcal{F}(U) \quad \text{w/ } \mathcal{F}(U) \text{ AN ABELIAN GROUP}$$

& "RESTRICTION MAPS" ρ_{UV} FOR EACH INCLUSION $V \hookrightarrow U$.
$\in$ GROUP HOMS

A MORPHISM OF SHEAVES IS A COLLECTION OF GROUP HOMS

$$\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

WHICH COMMUTE WITH RESTRICTION

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "\varphi_V"'] & \mathcal{G}(V). \end{tikzcd}$$

NOTE THAT THIS IS EXACTLY THE DEFINITION OF A NAT^{l} TRANS^{n} OF FUNCTORS.

CONSIDER THE PRESHEAF $\ker\varphi : U \mapsto \ker\varphi_U \subset \mathcal{F}(U)$
LETS PROVE THIS IS A SHEAF. FIRST, WE SHOW THAT
IF $\sigma \in (\ker\varphi)(U)$ IS $\ni$ $\sigma|_{U_i} = 0$, $\sigma = 0$.

THIS FOLLOWS IMMEDIATELY FROM THE FACT THAT
$(\ker\varphi)(U) \subset \mathcal{F}(U)$, & $\mathcal{F}(U)$ IS A SHEAF.

Next, if $\sigma_i|_{U_{ij}} = \sigma_j|_{U_{ij}}$ then $\exists\ \sigma \in \ker \varphi_U \ni \sigma|_{U_i} = \sigma_i$; i.e.
w.t.s. $\exists\ \sigma \in \mathcal{F}(U) \ni \varphi_U(\sigma) = 0$ & $\sigma|_{U_i} = \sigma_i \ \forall i$

We know since $\mathcal{F}$ is a sheaf, $\exists$ a $\sigma \ni \sigma|_{U_i} = \sigma_i$, &
$\varphi(\sigma)|_{U_i} = \varphi(\sigma_i) = 0$ by commutivity of

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\mathcal{F}(U_i) \arrow[r] & \mathcal{G}(U_i)
\end{tikzcd}$$

Since $\mathcal{G}$ is a sheaf, $\varphi(\sigma)|_{U_i} = 0 \Rightarrow \varphi(\sigma) = 0$, & thus
$\sigma \in \ker \varphi_U$.

Now, consider the presheaf $\operatorname{im}\varphi : U \longmapsto \operatorname{im}\varphi_U \subset \mathcal{G}(U)$
It is easy to see that if $\sigma \in \operatorname{im}\varphi_U$ &
$\sigma|_{U_i} = 0$, then $\sigma = 0$, since $\mathcal{G}$ is a sheaf.
The problem is that it is not guarenteed that $\sigma_i|_{U_{ij}} = \sigma_j|_{U_{ij}}$
gives a section in $(\operatorname{im}\varphi)(U)$. For sure, we can glue
to a section $\sigma \in \mathcal{G}(U)$, but there may not be a $\tau \in \mathcal{F}(U)$
s.t. $\varphi(\tau) = \sigma$.

Lets look at an example where this is not the case.

ASIDE

LET'S COVER S^1 WITH TWO OPEN SETS. WE THINK ABOUT S^1 AS ~~$0 \leq \theta \leq 2\pi$ WITH $2\pi \sim 0$.~~ A "COORDINATE" θ IDENTIFIED OVER 2π.

THEN, DEFINE TWO OPEN SETS

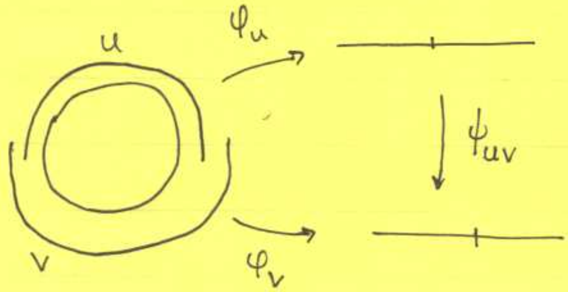
$$U = \{\theta \mid -\epsilon < \theta < \pi + \epsilon\}$$
$$V = \{\theta \mid \pi - \epsilon < \theta < 2\pi + \epsilon\}$$

AN ATLAS FOR A MANIFOLD M IS A COLL^n OF CHARTS $\{U_i, \varphi_i\}$, WHERE THE U_i COVER M & $\varphi_i : U_i \to \mathbb{R}^n$

THUS WE DEFINE $\varphi_u : U \to \mathbb{R}$ AS $\theta \mapsto \theta$

& $\varphi_v : V \to \mathbb{R}$, AS $\theta \mapsto \theta$

WE MUST ALSO SPECIFY TRANSITION FUNCTIONS ON $U \cap V$, WHICH TELL HOW TO GO BETWEEN CHARTS



THE ONLY REQ.^{t} IS THAT ψ_{uv} BE DIFFERENTIABLE & INVERTIBLE ON $\varphi_u(U \cap V)$

$$\psi_{uv} = \begin{cases} \theta & \theta > \pi/2 \\ 2\pi + \theta & \theta < \pi/2 \end{cases}$$

③

C^∞ fcts NON-ZERO

CONSIDER THE SHEAVES $\mathbb{Z}$, $\mathcal{O}$, & $\mathcal{O}^*$ ON THE CIRCLE.
COVER S^1 WITH OPEN SETS

$$\mathcal{U} = -\epsilon \leq \theta < \pi + \epsilon$$
$$V = \pi - \epsilon \leq \theta < 2\pi + \epsilon$$

FOR ANY OPEN $(X, U, V, U \cap V)$, WE DEFINE

$$i_u : \mathbb{Z}(U) \to \mathcal{O}(U)$$
$$i \mapsto i$$
$$e^{2\pi i} : \mathcal{O}(U) \to \mathcal{O}^*(U)$$
$$f \mapsto e^{2\pi i f}$$

CLEARLY, KER $e^{2\pi i} \cong \mathbb{Z}$. WE WOULD LIKE TO KNOW WHETHER OR NOT IM $e^{2\pi i}$ IS A SHEAF

CONSIDER THE C^∞ fcts $f_u : \theta \to \theta$
& $f_v : \theta \to \theta$

ON U & V. CLEARLY, $f_u|_{U\cap V} \neq f_v|_{U \cap V}$, ALTHOUGH

$$e^{2\pi i f_u}|_{U \cap V} = e^{2\pi i f_v}|_{U\cap V}.$$

THUS, WE HAVE SECTIONS IN $(\text{Im } e^{2\pi i})(U, V)$ WHICH DO ~~NOT~~ GLUE TO A SECTION of $C^*(S^1)$, BUT NOT A SECTION of $(\text{Im } e^{2\pi i})(S^1)$.

④

Now that we have seen that the image of a sheaf morphism may not be a sheaf, lets describe a way to fix the problem; Sheafification.

This is a procedure which produces a sheaf from a presheaf. If $\mathcal{F}$ is a sheaf, then sheafification of the underlying presheaf produces a sheaf isomorphic to $\mathcal{F}$.

Recall that the stalk of a sheaf $\mathcal{F}$ at a point p is the group of equivalence classes

$$\mathcal{F}_p = \frac{\{(U,s),\ U \text{ open } \& \ p\in U,\ s\in\mathcal{F}(U)\}}{(U,s)\sim(V,t)}$$

where $(U,s)\sim(V,t)$ if $\exists\ W\subset U\cap V$ such that $p\in W$ & $s|_W = t|_W$

Thus the stalk of a sheaf at p depends on the data around p, not just at p.

Now, let me define the coproduct, or disjoint union. Let U_i be open sets in X, w/ $\bigcap_i U_i \neq 0$

Then, their coproduct $\coprod_i U_i$ is the set

$$\bigcup_{i\in I}\{(x,i) : x\in U_i\}$$

Thus → : Even though $U_i\cap U_j \neq \phi$ in X, $U_i\cap U_j = \phi$ in $\coprod_i U_i$

NOW WE ARE ABOUT TO EMBARK ON A BEAUTIFUL CONSTRUCTION, THE ESPACE ÉTALÉ. ÉTALÉ IS A FRENCH WORD USED IN POETRY TO MEAN THE APPEARANCE OF A CALM SEA UNDER A FULL MOON; ONLY SLIGHT PERTURBATIONS ARE APPARENT. IT CAN ALSO MEAN "SLACK", OR "SPREAD OUT".

GIVEN A PRESHEAF $\mathcal{F}$, WE DEFINE ITS ~~ÉTALÉ~~ ÉTALÉ SPACE TO BE
$\downarrow$
X

$$\mathcal{E}_{\mathcal{F}} = \coprod_{x \in X} \mathcal{F}_x .$$

NOW, WE ARE GOING TO DEFINE A TOPOLOGY ON $\mathcal{E}_{\mathcal{F}}$ SO THAT CONTINUOUS MAPS $X \to \mathcal{E}_{\mathcal{F}}$ FORM A SHEAF.

RECALL THAT A BASIS $\mathcal{B}$ FOR A TOPOLOGY IS A COLLECTION OF SUBSETS SATISFYING

1) $\forall\, x \in \mathcal{E}_{\mathcal{F}}$, $\exists\, U \subset \mathcal{B}$ WITH $x \in U$
2) IF $U, V \in \mathcal{B}$ w/ $x \in U \cap V$, $\exists\, W \in \mathcal{B}$ WITH $W \subset U \cap V$

& THAT THE TOPOLOGY GEN^{d} BY A BASIS IS THE COLLECTION OF ALL ~~THE~~ UNIONS & FINITE INTERSECT^{n}S OF BASIS ELEMENTS.

DEFINE A COLLECTION OF 'BASIS ELEMENTS' AS FOLLOWS;

FOR EACH $U \subset X$, AND FOR EACH SECTION $s \in \mathcal{F}(U)$,

$$\mathcal{U}_s = \{ s_x ,\ x \in \mathcal{U} \}.$$

HERE: s_x IS THE GERM OF s AT x. WE VERIFY THAT THIS FORMS A BASIS.

~~LET s_x BE A POINT IN $\mathcal{E}_\mathcal{F}$.~~

~~(1) SINCE WE HAVE SOME COVER $\{U_i\}$ OF X TO DEFINE THE SHEAF, $\forall x \; \exists \; U \subset X$ WITH $x \in U$. FURTHERMORE, SINCE $\mathcal{F}(U)$ IS AN ABELIAN GROUP, IT IS NON-EMPTY, SO $\mathcal{U}_s \neq \emptyset$ FOR SOME $s \in \mathcal{F}(U)$.~~ THEN,

~~LET s_x BE A POINT~~ $\subset U$

(1) LET s_x BE A POINT IN $\mathcal{E}_\mathcal{F}$, & (U, t) BE A REP^VE OF s_x. THEN $\mathcal{U}_t = \{ t_y ,\ y \in U \}$ IS A SET CONTAINING s_x

(2) IF $s_x \in \mathcal{U}_t$ & $\mathcal{V}_v$; (U,t) & (V,v) A REPS OF s_x. SINCE THEY ARE EQUIVALENT, $\exists \; W \subset U \cap V$ WITH $w = t|_W = v|_W$. THEN $\mathcal{W}_w \subset \mathcal{U}_t \cap \mathcal{V}_v$ & $s_x \in \mathcal{W}_w$

THEN WE DEFINE A NEW SHEAF $\tilde{\mathcal{F}}$ WHOSE SECTIONS ARE CONTINUOUS FUNCTIONS $U \mapsto \mathcal{E}_\mathcal{F}$.

ROUGHLY, WE ARE TAKING ALL MAPS TO STALKS & IDENTIFYING THINGS WHICH HAVE THE SAME RESTRICTIONS & ADDING THINGS THAT GLUE.

(7)

NOW, WE VERIFY THAT $\tilde{\mathcal{F}}$ IS A SHEAF,
SHOW THAT $\tilde{\mathcal{F}} \simeq \mathcal{F}$ IF $\mathcal{F}$ IS A SHEAF,
& EXPLAIN WHAT'S GOING ON.

WE MUST SHOW THAT $\tilde{\mathcal{F}}$ SATISFIES (3) & (4). LET $\mathcal{U} = \bigcup_i \mathcal{U}_i \subset X$.

(3) IF $\sigma_i \in \tilde{\mathcal{F}}(\mathcal{U}_i)$ & $\sigma_j \in \tilde{\mathcal{F}}(\mathcal{U}_j)$ W/ $\sigma_i|_{\mathcal{U}_{ij}} = \sigma_j|_{\mathcal{U}_{ij}}$ THEN $\exists\, \sigma \in \tilde{\mathcal{F}}(\mathcal{U})$ W/ $\sigma|_{\mathcal{U}_i} = \sigma_i$

THE σ_j ARE CONTINUOUS FUNCTIONS $\mathcal{U}_j \to \coprod_{x \in \mathcal{U}_j} \mathcal{F}_x$. WE WANT TO SHOW THEY PATCH TO A CONTINUOUS FUNCTION $\sigma : \mathcal{U} \to \coprod_{x \in \mathcal{U}} \mathcal{F}_x$.

DEFINE $\sigma(x) = \sigma_i(x)$ IF $x \in \mathcal{U}_i$. NOTE THAT IF $x \in \mathcal{U}_j$,
$= s_x \in \mathcal{F}_x$

$$\sigma(x) = \sigma_j(x) = \sigma_i(x) = s_x \in \mathcal{F}_x$$

SO THE FUNCTION MAKES SENSE PATCH-WISE. WE NOW CHECK THAT σ IS CONTINUOUS. CONSIDER AN OPEN SET

$$\mathcal{V}_v = \{ v_x ,\ x \in V \} \subset \coprod_{x \in \mathcal{U}} \mathcal{F}_x \quad \& \quad V \subset \mathcal{U}.$$

THEN CLEARLY $\sigma^{-1}(\mathcal{V}_v) = V$ IS OPEN IN X.

(4) IF $\sigma \in \tilde{\mathcal{F}}(\mathcal{U})$ & $\sigma|_{\mathcal{U}_i} = 0 \ \forall i$, THEN $\sigma = 0$.

THIS FOLLOWS IMMEDIATELY FROM THE DEFINITION.

(8)

Let's construct a 1-1 onto function $\mathcal{F}(U) \to \tilde{\mathcal{F}}(U)$ for each $U \subset X$ open.

Let $s \in \mathcal{F}(U)$ & define a function $\tilde{s}: U \to E_{\mathcal{F}}$ by

$$\tilde{s} : x \mapsto s_x .$$

We check that $\tilde{s}$ is continuous. Clearly, $\operatorname{Im} \tilde{s} = \mathcal{U}_s$. We want to show that for any $\mathcal{V}_\sigma$ with $\mathcal{V}_\sigma \cap \mathcal{U}_s \neq \emptyset$, $\tilde{s}^{-1}(\mathcal{V}_\sigma \cap \mathcal{U}_s)$ is open. We assume that $V \subset U$. Then, $\tilde{s}^{-1}(\mathcal{V}_\sigma \cap \mathcal{U}_s) = \{ x \in U \mid \sigma_x = s_x \}$

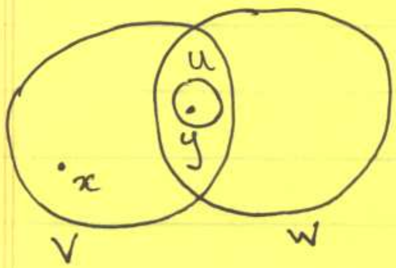
Two germs are equal when they match on a small nbhd; for some $W \subset U \cap V$ w/ $x \in W$, $\sigma|_W = s|_W$.
Since we can find such a W for each x, we take their union & obtain an open set in U where σ & s match.

Thus, $\forall\, s \in \mathcal{F}(U)$ $\exists$ a cts $\tilde{s} \in \tilde{\mathcal{F}}(U)$. Now we must show that if $\tilde{s} = \tilde{t}$, $s = t$; i.e. sections are equal if they match at each stalk. But this follows by the same reasoning as before; in fact we construct an open cover of U where s & t are equal on each set, so they are equal on all of U.

Now, we want to show that for any cts fct^n $\tilde{s} : U \to E_{\mathcal{F}}$, $\exists$ $s' \in \mathcal{F}(U)$ with $\tilde{s}' = \tilde{s}$. Take an open cover $\{\mathcal{V}_{s_i}\}$ of the image of $\tilde{s}$. If $\mathcal{V}_{s_i} \cap \mathcal{V}_{s_j} \neq \emptyset$, then on $V_i \cap V_j$, $s_i|_{V_{ij}} = s_j|_{V_{ij}}$ & since $\mathcal{F}$ is a sheaf, $\exists!$ $s_{ij} \in \mathcal{F}(V_i \cup V_j)$, which restricts to each V_i resp. Repeating, we glue to a section $s \in \mathcal{F}(U)$.

LET'S GET AN IDEA OF WHAT'S GOING ON. THE SHEAFIFICATION "WORKS" BECAUSE FUNCTIONS MAY ALWAYS BE PATCHED TOGETHER.

CONSIDER THE STALK $\mathcal{F}_x$ & AN ELEMENT $s_x \in \mathcal{F}_x$. CHOOSE SOME REP OF s_x, SAY (V, s). THEN $\exists$ A CTS FCT^n $\sigma : V \to \mathcal{E}_{\mathcal{F}}$ WITH $\sigma(x) = s_x \ \forall \ x \in V$.



CONSIDER ANOTHER POINT $y \in V$ & THE GERM s_y. IT HAS ANOTHER REP (W, t), & $\exists$ $U \subset V \cap W$ WITH $s|_U = t|_U$

SO, WE HAVE CONTINUOUS FUNCTIONS WHICH MATCH ON A OPEN SET & MAY THUS BE EXTENDED TO A CTS FUNCTION ON THE UNION.

10

LETS REËXAMINE THE IMAGE PRESHEAF & ITS ASSOCIATED SHEAF.

THE PROBLEM WE ENCOUNTERED IN OUR EXAMPLE WAS THAT UNDER THE EXPONENTIAL MAP,

$$f_U : \theta \longmapsto \theta$$
$$f_V : \theta \longmapsto \theta$$

MAP TO A SECTION WHICH IS NOT IN THE IMAGE OF f_{S^1}.

THE STALK OF $\mathcal{O}^*$ AT ANY POINT IS $\mathbb{C}^*$, SO WE ARE LOOKING AT FUNCTIONS $U \mapsto \coprod_{x \in U} \mathbb{C}^*$, WHICH ARE CONTINUOUS. THE TOPOLOGY ON $\coprod_{x \in U} \mathbb{C}^*$ IS SETS OF THE FORM (U, s), WITH $s : U \to \mathbb{C}^*$.

THEN A CTS FCT $U \mapsto \coprod_{x \in U} \mathbb{C}^*$ IS EXACTLY ONE OF THESE s's

THE DIFFERENCE IS THAT WHEN WE EXAMINE $U \cup V$, WE HAVE MORE FCTS, SINCE WE CAN HAVE s_U & t_V WHICH MATCH ON $U \cap V$.

A FCT $f : S^1 \to \coprod_{x \in S^1} \mathbb{C}^*$

$f(\theta) = e^{i\theta}$ IS CTS, SINCE THE OPEN SETS ARE

$$(U, s_U) \quad (V, t_V) \quad (S^1, r_{S^1})$$

$$\& \quad f^{-1}(S^1\, r_{S^1}) = \phi$$
$$f^{-1}(U\, s_U) = U \qquad \text{if } s_U = e^{i\theta}$$
$$f^{-1}(V\, t_V) = V \qquad \text{if } t_V = e^{i\theta}$$