①

Factor Rings:

Consider a ring R & an ideal $\mathcal{J} \subset R$. Define the factor ring $R/\mathcal{J}$ as the cosets of $\mathcal{J}$ as a subgroup (the set of subsets of the form $(r+\mathcal{J})\ \forall\, r \in R$), with multiplication defined as $(r+\mathcal{J})(s+\mathcal{J}) = (rs+\mathcal{J})$

We show that mult^n is well def^d on cosets; if r, s & r', s' are in the same cosets, then so are $r \cdot s$ & $r's'$.

Same coset $\Rightarrow r = r' + \mathfrak{r}$, $s = s' + \mathfrak{s}$.

So $rs = r's' + \underbrace{r'\mathfrak{s} + s'\mathfrak{r} + \mathfrak{s}\mathfrak{r}}_{\in \mathcal{J}}$

$\Rightarrow rs$ & $r's'$ in same coset.

Thus, $R/\mathcal{J}$ is a ring, & we have a canonical ring homomorphism $\pi : R \to R/\mathcal{J}$

Example; consider $j \in \mathbb{Z}$, $j \geq 0$, & the ideal $j\mathbb{Z} \subset \mathbb{Z}$. We construct the factor ring $\mathbb{Z}/j\mathbb{Z}$ to be the sets of integers differing by j;

$(-1 + j\mathbb{Z}) = \{\cdots, -1, j-1, 2j-1, \cdots\}$

$(0 + j\mathbb{Z}) = \{\cdots, -j, 0, j, \cdots\}$

$(1 + j\mathbb{Z}) = \{\cdots, -j+1, 1, 1+j, \cdots\}$

$\vdots$

$(j + j\mathbb{Z}) = \{\cdots, j-j, j+0, j+j, \cdots\}$

~~One can think of a factor ring as "setting the ideal to zero"~~

②

ONE CAN IMMEDIATELY SEE THAT $\forall\ k \in \mathbb{Z},\ (k, j\mathbb{Z}) \sim (k+j, j\mathbb{Z})$, SO THAT THERE ARE EXACTLY j UNIQUE COSETS. ~~CANONICAL~~ THEY ARE USUALLY WRITTEN AS STARTING WITH ZERO, & WE DEFINE

$$\mathbb{Z}_j = \mathbb{Z}/\mathbb{Z}_j = \{(0+j\mathbb{Z}), \cdots, (j-1, j\mathbb{Z})\}$$

OR MORE SIMPLY $\mathbb{Z}_j = (\mathbb{Z} \text{ MODULO } j)$

EX; $\mathbb{Z}_6 = \{0, 1, 2, 3\ 4\ 5\}$

$$\text{WITH} \quad 4+5 = 3 \bmod 6$$
$$\& \quad 3 \cdot 4 = 0 \bmod 6$$

THIS BRINGS US TO OUR NEXT ~~EXAMPLE~~ DEFINITION, ZERO DIVISORS. ~~A~~ ZERO DIVISORS $x\,y$ ~~a~~ $\in R$ ~~ARE~~ ARE NON-ZERO ELEMENTS S.T. ~~B~~ $x \cdot y = 0$. IN $\mathbb{Z}_6$, WE SEE THAT $\{2, 3, 4\}$ ARE ZERO DIVISORS.

~~Co~~

EX; CONSIDER $\mathbb{Z} \times \mathbb{Z}$; $(x,y) + (z,w) = (x+z, y+w)$
$(x,y) \cdot (z,w) = (x \cdot z, y \cdot w)$.

NOTE THAT ~~ALL~~ $\forall$ INTEGERS $j \in \mathbb{Z}$, $(j, 0)$ & $(0, j)$ ARE ZERO DIVISORS.

RECALL OUR DISCUSSION of PRIME IDEALS. IF R IS A RING WITH ZERO DIVISORS, $\{(0)\}$ IS <u>NOT</u> PRIME.

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ANOTHER IMPORTANT TYPE OF IDEALS ARE MAXIMAL IDEALS. AN IDEAL $\mathfrak{m} \subset R$ IS MAXIMAL IF $\mathfrak{m} \neq R$ & $\nexists$ AN IDEAL $\mathfrak{p} \subset R$ S.T. $\mathfrak{p} \subset \mathfrak{m}$ & $\mathfrak{p} \neq \mathfrak{m}$.

AN IMMEDIATE CONSEQUENCE IS THAT EVERY MAXIMAL IDEAL IS PRIME; LET $\mathfrak{m}$ BE MAX^l & $x, y \in R$ WITH $x \cdot y \in \mathfrak{m}$. $\mathfrak{m} + x \cdot R$ IS AN IDEAL OF THE RING PROPERLY CONTAINING $\mathfrak{m}$ => $\mathfrak{m} + x \cdot R = R$.

(1) $\mathfrak{m} + \underset{(x)}{x \cdot R}$ IS AN IDEAL;

$$(m + xr) + (n + xs) = (m+n) + x(r+s)$$

$$\& \quad r(m + x \cdot s) = rm + x \cdot rs \in \mathfrak{m} + (x).$$

THUS $\exists$ $r \in R$ & $m \in \mathfrak{m}$ S.T. $1 = m + x \cdot r$

$$y = y \cdot m + yx \cdot r$$

$$= \underset{\mathfrak{m}}{y \cdot m} + \underset{\mathfrak{m}}{r \cdot (xy)}$$

& SINCE $\mathfrak{m}$ IS AN ADDITIVE SUBGROUP OF R, $y \in \mathfrak{m}$.

RECALL THAT A FIELD IS A RING WHERE THE GROUP OF UNITS $U = R$. IT TURNS OUT THAT THE ONLY IDEALS OF A FIELD ARE $\{0\}$ & THE FIELD ITSELF.

LET K BE A FIELD & $I \subseteq K$ AN IDEAL W/ $I \neq \{0\}$. THEN IF $r \in I$, $r^{-1} \cdot r \in I$, BUT $r^{-1} \cdot r = 1$, SO $1 \in I = K$.
$(\forall k \in K, \ 1 \cdot k \in I \Rightarrow I = K)$

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If $\mathfrak{M}$ IS A MAXIMAL IDEAL IN R, $R/\mathfrak{M}$ IS A field.

Pf: RECALL WE HAVE THE NATURAL HOM^m $f: R \to R/\mathfrak{M}$, $f(x) = (x + \mathfrak{M})$. DENOTE $\bar{x} = f(x)$, & ASSUME $x \neq 0 \neq \bar{x}$. RECALL THAT IN THE COURSE of OUR proof THAT MAX^l IDEALS ARE PRIME, WE SHOWED THAT

$$\mathfrak{M} + R \cdot x = R$$

$(\bar{x} \neq 0 \Rightarrow x \notin \mathfrak{M})$. FURTHERMORE, WE SHOWED THAT

$$1 = u + y \cdot x$$

WITH $u \in \mathfrak{M}$ & $y \in R$. Applying f ON BOTH SIDES, WE find THAT

$$1 = \bar{y} \cdot \bar{x},$$

SO, EACH NON-ZERO ELEMENT of $R/\mathfrak{M}$ HAS ~~AN~~ MULTIPLICATIVE INVERSE.

LET $f: R \to S$ BE A RING HOM^m. If $\mathfrak{p}$ IS A PRIME IDEAL of S, $f^{-1}(\mathfrak{p})$ IS A PRIME IDEAL of R.

Pf: WE MUST SHOW THAT if x & $y \in R$ & $x \cdot y \in f^{-1}(\mathfrak{p})$, THEN $x \in f^{-1}(\mathfrak{p})$ OR $y \in f^{-1}(\mathfrak{p})$. BUT THIS follows IMMEDIATELY from THE fact f IS A RING HOM.

$x \cdot y \in f^{-1}(\mathfrak{p}) \Rightarrow f(x) \cdot f(y) \in \mathfrak{p}$.

$\mathfrak{p}$ prime implies $f(x)$ or $f(y) \in \mathfrak{p}$ (SAY ITS $f(x)$).

THEN $x \in f^{-1}(\mathfrak{p})$.

⑤

WE NOW MOVE TO LOCALIZATION IN A RING. THIS IS THE METHOD of "INVERTING" ELEMENTS.

A BRIEF ASIDE: WHEN WE THINK of DIVISION, WHAT WE ARE REALLY THINKING ABOUT IS MULTIPLICATION BY THE MULTIPLICATIVE INVERSE; $\frac{x}{y} = y^{-1}\cdot x$. NOTE THAT THIS IS ONLY MEANINGFUL (OR WELL-DEFINED) if $y^{-1}\cdot x = x\cdot y^{-1}$ $\forall$ x, y.

Of COURSE, IN ANY GIVEN RING, MOST ELEMENTS DO NOT HAVE A MULTIPLICATIVE INVERSE; $\nexists$ AN INTEGER j S.T. $2j = 1$. THUS, TO "INVERT" ELEMENTS of THE RING, WE CONSTRUCT A NEW RING, THE QUOTIENT RING (NOT THE FACTOR RING).

FIRST ~~WE DEFINE~~ LET $S \subset R$ BE A SUBSET CONTAINING 1 & S.T. $\forall$ $s, s' \in S$, $s \cdot s' \in S$. THEN THE QUOTIENT RING, OR RING of FRACTIONS of R BY S, IS THE SET of EQUIVALENCE CLASSES of THE form ($r', r \in R$, $s, s' \in S$)

$$(r, s) \sim (r', s')$$

WHERE $(r', s') \sim (r, s)$ if $\exists$ AN $\Delta \in S$ SUCH THAT

$$\Delta(s'r - r's) = 0$$

AN EQUIVALENCE CLASS IS DENOTED BY $\frac{r}{s}$ & THE SET of CLASSES IS WRITTEN $S^{-1}R$. NOTE; if $s \in S$, s^{-1} MAY NOT BE... IN FACT WE CAN TAKE $S = (0, 1)$

Then, we define $(r,s) \sim (r',s')$ if $\exists\, s \in S$ with

$$s(r's - rs') = 0$$

However since $0 \in S$, we can always take $s = 0$, so that any ~~two~~ pair ~~are~~ is equivalent to any other. Thus, the only element of $S^{-1}R$ when S contains 0 is $\frac{0}{1}$.

Consider ~~Indeed~~ $S = (1,3)$ in the ring $\mathbb{Z}_6$.

$$3 \cdot 3 = 9 \bmod 6 = 3$$

So S is multiplicatively-closed. Note that ~~if we take~~ we have that

$$(1,3) \sim (1,1)$$

since $3 \cdot (3 \cdot 1 - 1 \cdot 1) = 3(2) = 6 = 0 \bmod 6$. So in $S^{-1}\mathbb{Z}_6$, we have that $\frac{1}{3} = \frac{1}{1}$, & $\frac{2}{3} \cdot 3 \equiv \frac{6}{3} = 0$

The moral of this example is that when a ring has zero divisors, the quotient ring may not be the usual fractions.

In any case, we can make $S^{-1}R$ a ring with the definitions $\left(\frac{r}{s}\right)\left(\frac{r'}{s'}\right) = \frac{r \cdot r'}{s \cdot s'}$, $\frac{1}{1}$ the unit element, & $\frac{r}{s} + \frac{r'}{s'} \equiv \frac{rs' + r's}{s \cdot s'}$

Lets verify that addition is well defined: if $(r_1, s_1) \sim (r_1', s_1')$ & $(r\,s) \sim (r', s')$, we must show that

$$\frac{rs_1 + r_1 s}{s_1 s} = \frac{r}{s} + \frac{r_1}{s_1} \sim \frac{r'}{s'} + \frac{r_1'}{s_1'} = \frac{r's_1' + r_1's'}{s's_1'}$$

Let $s_2 \in S$ be s.t. $s_2(rs' - r's) = 0$

& $s_2' \in S$ be s.t. $s_2'(r_1 s_1' - r_1' s_1) = 0$

~~MULTIPLYING BY~~ $\cancel{s_2 s_1 s}$ ~~&~~ $\cancel{s_3 s' s_1'}$

WE VERIFY THAT ADDITION IS WELL-DEFINED W.R.T. THE EQUIVALENCE CLASSES;

TAKE $\frac{r_1}{s_1} = \frac{r}{s}$ & $\frac{r_1'}{s_1'} = \frac{r'}{s'}$. THUS $\exists$ $s_2, s_2' \in S$ SUCH THAT

$$s_2(r_1 s - r s_1) = 0$$
$$s_2'(r_1' s' - r' s_1') = 0.$$

WTS. $\frac{r_1}{s_1} + \frac{r_1'}{s_1'} = \frac{r}{s} + \frac{r'}{s'} \rightsquigarrow \frac{r_1 s_1' + s_1 r_1'}{s_1 s_1'} = \frac{r s' + r' s}{s s'}$.

MULTIPLYING BY $s_2' s' s_1'$ & $s_2 s s_1$, WE HAVE

$$s_2 s_2' \left(s' s_1' (s r_1 - s_1 r) + s s_1 (s' r - s_1' r') \right) = 0$$

$$s_2 s_2' \left(s s' (s_1' r_1 + s_1 r_1') - s_1 s_1' (s' r + s r') \right) = 0$$

Q.E.D.

THUS, $S^{-1}R$ IS A RING, & WE ALWAYS HAVE A HOM^m^ $\phi: R \to S^{-1}R$ DEF^d^ AS $\varphi(r) = \frac{r}{1}$.

If R HAS NO ZERO-DIVISORS, THEN $S = R - \{0\}$ IS CLOSED UNDER MULTIPLICATION, & $S^{-1}R$ IS A *field*, CALLED THE <u>field of fractions</u> of R.

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EXAMPLES; $R = \mathbb{Z}$, $S = \mathbb{Z} - \{0\}$. THEN $S^{-1}R \simeq \mathbb{Q}$, A *field*.
$R = \mathbb{R}$ $S = \mathbb{R} - \{0\}$; $S^{-1}\mathbb{R} \cong \mathbb{R}$

INDEED, if k IS ANY field, ITS field of fractions IS ISOMORPHIC TO k BY THE NATURAL MAP $\varphi(x) = \frac{x}{1}$.

$R = \mathbb{C}[x]$ $S = \mathbb{C}[x] - \{0\}$
$S^{-1}R \equiv K[x]$, <u>RATIONAL</u> *functions*

A RATIONAL function IS SOMETHING of THE form $f(x)/g(x)$ w/ $f, g \in \mathbb{C}[x]$.

LOCALIZATION AT A PRIME IDEAL.

LET $\mathfrak{p}$ BE PRIME IN A RING R
THRM; $R \setminus \mathfrak{p}$ CONTAINS 1 & IS MULT·PLICATIVELY CLOSED.
Pf; $\mathfrak{p}$ IS PRIME, SO IS NOT $= R$ & THUS DOES NOT CONTAIN 1.
WTS if x & $y \notin \mathfrak{p}$ THEN $x \cdot y \notin \mathfrak{p}$.
for A PRIME, if $x \cdot y \in \mathfrak{p}$ THEN $x \in \mathfrak{p}$ OR $y \in \mathfrak{p}$.
– OPPOSITE; if $x \notin \mathfrak{p}$ <u>AND</u> $y \notin \mathfrak{p}$, $x \cdot y \notin \mathfrak{p}$

THUS WE CAN TAKE $S = R \setminus \mathfrak{p}$ & DEfINE THE LOCALIZATION of A RING AT A PRIME IDEAL TO BE $R_{\mathfrak{p}} \equiv \frac{R}{R \setminus \mathfrak{p}}$.

~~FINALLY, A LOCAL RING IS ONE A RING WITH A UNIQUE MAXIMAL IDEAL. AN EXAMPLE IS $R_{\mathfrak{p}}$, WHICH WHOSE MAXIMAL IDEAL CONSISTS of ELEMENTS $\frac{a}{p}$ WITH $a \in \mathfrak{p}$ & $z \in R \setminus \mathfrak{p}$.~~

A LOCAL RING IS ONE WITH A UNIQUE (ONLY ONE) MAXIMAL IDEAL. IN A LOCAL RING, THE SUM of TWO NON-UNITS IS A NON-UNIT & THE MAX^l IDEAL IS EXACTLY THIS IDEAL.

Example:
- fields; THE ONLY NON-UNIT IS (0) SO IT IS THE UNIQUE MAX^l IDEAL
- if $\mathfrak{p} \subset R$ is prime, $R_{\mathfrak{p}}$ is LOCAL; THE NON-UNITS ARE THOSE ELEMENTS of THE form $\frac{p}{r}$, WITH $p \in \mathfrak{p}$ & $r \in R \setminus \mathfrak{p}$. ($p^{-1} \notin R$ SINCE THEN $p^{-1} \cdot p = 1 \in \mathfrak{p}$, WHICH CANNOT BE) (THIS IS WRITTEN $\mathfrak{p} R_{\mathfrak{p}}$.)