LAST TIME, WE FOUND THAT A PERTURBATIVE ANALYSIS CORRECTLY CALCULATED THE WITTEN INDEX, BUT GAVE TOO MANY GROUND STATES.

TO BEGIN OUR DISCUSSION OF NON-PERTURBATIVE EFFECTS, LET US GENERALIZE OUR DISCUSSION OF POTENTIAL THEORY TO THE SUPERSYMMETRIC σ-MODEL ON A RIEMANNIAN MANIFOLD M. THE POTENTIAL WILL BE A FUNCTION

$$h : M \longrightarrow \mathbb{R}.$$

THE σ-MODEL LAGRANGIAN IS MODIFIED BY THE TERMS

$$\Delta L = -\frac{1}{2} g^{IJ} \partial_I h \, \partial_J h - D_I \partial_J h \, \bar{\psi}^I \psi^J$$

WHERE $D_I \partial_J h = \partial_I \partial_J h - \Gamma^K_{IJ} \partial_K h$.

THE FULL LAGRANGIAN IS INVARIANT UNDER THE SUSY

$$\begin{aligned}
\delta \phi^I &= \epsilon \bar{\psi}^I - \bar{\epsilon} \psi^I \\
\delta \psi^I &= \epsilon \left(i \dot{\phi}^I - \Gamma^I_{JK} \bar{\psi}^J \psi^K + g^{IJ} \partial_J h \right) \\
\delta \bar{\psi}^I &= \bar{\epsilon} \left(-i \dot{\phi}^I - \Gamma^I_{JK} \bar{\psi}^J \psi^K + g^{IJ} \partial_J h \right)
\end{aligned}$$

$$L_0 = \frac{1}{2} g_{IJ} \dot{\phi}^I \dot{\phi}^J + \frac{i}{2} g_{IJ} \left(\bar{\psi}^I D_t \psi - D_t \bar{\psi} \psi \right) - \frac{1}{2} R_{IJKL} \psi^I \bar{\psi}^J \psi^K \bar{\psi}^L$$

THE NÖTHER CHARGES ARE

$$Q = \bar{\psi}^I (i p_I + \partial_I h)$$
$$\bar{Q} = \psi^I (-i p_I + \partial_I h).$$

FERMION PHASE ROTATION SYMMETRY AGAIN YIELDS THE CHARGE $F = g^{IJ} \bar{\psi}^I \psi^J$. THE CANONICAL COMMUTATION RELATIONS ARE ALSO UNCHANGED, SO THE HILBERT SPACE IS AGAIN $\mathcal{H} = \Omega^{\bullet}(\mathcal{M}) \otimes \mathbb{C}$. THE NÖTHER CHARGES ARE REPRESENTED AS

$$\begin{aligned} Q &= d + (d\phi^I \wedge) \partial_I h \\ &= d + dh \wedge \\ &= e^{-h} d e^{h} \equiv d_h \\ \bar{Q} &= Q^{\dagger} \\ &= e^{h} d^{\dagger} e^{-h} \equiv d_h^{\dagger}. \end{aligned}$$

WE AGAIN CHOOSE THE OPERATOR ORDERING IN THE HAMILTONIAN SO THAT

$$\{Q, \bar{Q}\} = 2H,$$

$$H = \tfrac{1}{2}\left(d_h d_h^{\dagger} + d_h^{\dagger} d_h\right).$$

FURTHUR, THE SPACE of SUSY GROUND STATES IS ISOMORPHIC TO THE COHOMOLOGY of Q. INDEED, THE COMPLEX of Q IS ISOMORPHIC TO THE UNDEFORMED COMPLEX;

$$Q = e^{-h} Q_{h=0} e^{h},$$

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^i(M) \arrow[r, "d"] \arrow[d, "e^{-h}"'] & \Omega^{i+1}(M) \arrow[r] \arrow[d, "e^{-h}"] & \cdots \\
\cdots \arrow[r] & \Omega^i(M) \arrow[r, "d_h"] & \Omega^{i+1}(M) \arrow[r] & \cdots
\end{tikzcd},$$

AND SO $\mathcal{H}^p_{(0)} = H^p(Q) \cong H^p(Q_{h=0}) \cong H^p_{DR}(M)$. WE SEE THAT THE WITTEN INDEX IS INDEPENDANT of h.

NOW, LETS ANALYZE THIS THEORY WITH THE AID of PERTURBATION THEORY. AGAIN, WE DEFORM BY

$$h \to \lambda h, \quad \lambda \gg 1.$$

NEAR A CRITICAL POINT x_i, WE EXPAND h;

$$h(x) = h(x_i) + \sum_{I=1}^{m} c_I (x^I)^2 + \cdots,$$

WHERE THE COORDINATES x^I HAVE BEEN CHOSEN TO DIAGNOLIZE THE HESSIAN AT x_i.

CLOSE TO A CRITICAL POINT x_i, TO FIRST ORDER IN PERTURBATION THEORY, THE HAMILTONIAN IS

$$H_0 = \sum_I \left\{ \frac{1}{2} P_I^2 + \frac{\lambda^2}{2} C_I^2 (x^I)^2 + \frac{\lambda}{2} C_I [\bar{\psi}^I, \psi^I] \right\},$$

AND TO LEADING ORDER, THE GROUND STATE AT x_i IS

$$\Psi_i^{(0)} = e^{-\lambda \sum_{I=1}^{N} |C_I| (x^I)^2} \prod_{J:\, C_J < 0} \bar{\psi}^J |0\rangle.$$

AGAIN, WE HAVE THAT $\Psi_i^{(0)}$ IS REPRESENTED BY A μ_i-FORM. IN FACT, HIGHER ORDER TERMS IN THE PERTURBATION EXPANSION DO NOT ALTER THE FORM DEGREE, SO THAT

$$\Psi_i \in \Omega^{\mu_i}(M) \otimes \mathbb{C}.$$

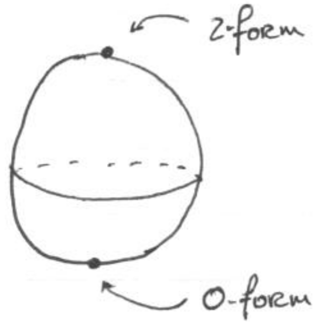
NOW, ALTHOUGH IT IS TRUE THAT $\langle \Psi^i | Q | \Psi^i \rangle = 0$ TO ALL ORDERS, IT MAY NOT BE TRUE THAT $\langle \Psi^i | Q | \Psi^j \rangle = 0$ FOR $i \neq j$.

~~Last time, we found that a perturbative analysis correctly calculated the Witten index, but gave too many ground states.~~

The problem with our perturbative analysis is that everything was calculated in terms of local data at the critical points. One cannot tell if the manifold's topology requires the critical point, or if it is "removable".

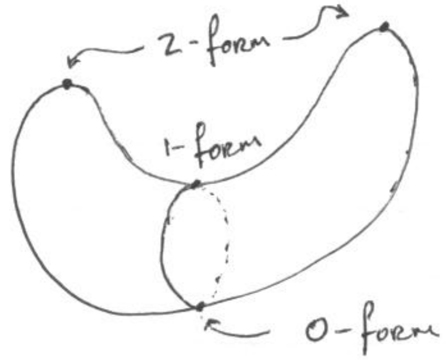
For example, consider $M = S^2$, with h as the height function.

(I)



We see there are two bosonic ground states in perturbation theory.

(II)



However, if we deform the sphere, the number of perturbative grd states is four. The full theory's ground state(s) cannot depend on the choice of h, so some ~~of~~ (or all) of these states are not the true ground states. In case (I), we cannot lift both bosonic states to a higher energy because of pair lifting. Thus, we see that there are

TWO SUSY GROUND STATES IN THE full THEORY.

NOW, WE SHALL LOOK FOR AN EXPANSION

$$Q|\Phi_i\rangle = \sum_{j=1}^{N} |\Phi_j\rangle\langle\Phi_j|Q|\Phi_i\rangle + \cdots .$$

THE $(\cdots)$ TERMS HAVE ENERGIES of $\mathcal{O}(\lambda)$, AND SO ARE SUPPRESSED BY factors of $\frac{1}{\lambda}$. THUS, WE NEED TO COMPUTE $\langle\Phi_i|Q|\Phi_j\rangle$. ONE THING WE CAN IMMEDIATLY SEE IS THAT

$$\langle\Phi_i|Q|\Phi_j\rangle = \int_{\mathcal{M}} \overline{\Phi_i} \wedge * \, d_h \Phi_j$$

WHICH VANISHES UNLESS $\mu_i = \mu_j + 1$.

NOW, WE ARE left WITH THE TASK of EVALUATING THIS EXPRESSION.

$$\langle\Phi_j|Q\,\Phi_i\rangle = \frac{1}{h(x_i) - h(x_j)} \lim_{T\to\infty} \langle\Phi_j|e^{-TH}[Q,h]e^{-TH}|\Phi_i\rangle$$

$$\lim_{T\to\infty} \quad \langle\Phi_j|[Q\,h(x) - h(x)\,Q]|\Phi_i\rangle$$

$$\langle\Phi_j|Q|\Phi_i\rangle\,(h(x_i) - h(x_j))$$

$$\begin{aligned}[Q,h] &= d_h h - h\, d_h \\ &= (d h)\wedge \\ &= \psi^I \partial_I h\end{aligned}$$

$$S_E = \int_{-\infty}^{\infty} d\tau \left\{ \frac{1}{2} g_{IJ} \frac{d\phi^I}{d\tau} \frac{d\phi^J}{d\tau} + \frac{\lambda^2}{2} g^{IJ} \partial_I h \partial_J h + g_{IJ} \bar{\psi}^I D_t \psi^J + \lambda D_I \partial_J h \, \bar{\psi}^I \psi^J + \frac{1}{2} R_{IJKL} \psi^I \bar{\psi}^J \psi^K \psi^L \right\} \quad (7)$$

WE HAVE THAT

$$[h(x_i) - h(x_j)] \langle \Psi_i | Q \Psi_j \rangle = \int_{\substack{\phi(-\infty)=x_i \\ \phi(+\infty)=x_j}} D\phi D\psi D\bar{\psi} \, e^{-S_E} \, \bar{\psi}^I \partial_I h(\tau=0)$$

THE BOSONIC PART OF THE ACTION IS

$$= \int_{-\infty}^{\infty} d\tau \left\{ \frac{1}{2} \left[\frac{d\phi^I}{d\tau} \pm \lambda g^{IJ} \partial_J h \right]^2 \mp \lambda \frac{d\phi^I}{d\tau} \partial_I h \right\}$$

$$= \text{———} '' \text{———} \mp \lambda \int dh$$

$$= \text{———} '' \text{———} \mp \lambda \left(h(x_j) - h(x_i) \right)$$

THUS, IF $h(x_j) > h(x_i)$, $\frac{d\phi^I}{d\tau} = \lambda g^{IJ} \partial_J h$, THE PATH OF STEEPEST ASCENT, MINIMIZES S_E.

IF $h(x_j) < h(x_i)$, $\frac{d\phi^I}{d\tau} = -\lambda g^{IJ} \partial_J h$, i.e. THE PATH OF STEEPEST DESCENT MINIMIZES S_E.

SUCH A ϕ IS CALLED AN INSTANTON. NOTE THAT AN INSTANTON MAY BE DEFORMED INTO ANOTHER BY TIME TRANSLATION $\phi'(\tau) = \phi(\tau + \delta\tau)$. ARE THERE OTHER DEFORMATIONS? THE FIRST ORDER VARIATION OF THE INSTANTON EQ. IS

$$D_\pm \delta\phi^I := D_\tau \delta\phi^I \pm \lambda g^{IJ} D_J \partial_K h \, \delta\phi^K = 0.$$

THE DIMENSION OF THE KERNEL OF THIS OPERATOR WILL TELL US THE NUMBER OF DEFORMATIONS.

SINCE $S_{\psi\bar\psi} = \int_{-\infty}^{\infty} d\tau\, g_{IJ} \bar\psi^I D_+ \psi^J = -\int_{-\infty}^{\infty} d\tau\, g_{IJ} D_- \bar\psi^I \psi^J$, AND SINCE THERE IS AN INSERTION OF A D_- ZERO MODE, FOR THE P.I. TO BE NONVANISHING, WE NEED FOR THE NUMBER OF $\bar\psi$ ZERO MODES TO BE GREATER THAN THE # OF ψ ZERO MODES BY ONE:

$$\operatorname{IND} D_- = 1.$$

NOW IF WE MAKE THE ASSUMPTION THAT $\ker D_+ = 0$, FOR INSTANTONS, WE WILL HAVE THAT THE ONLY DEFORMATION POSSIBLE IS A SHIFT IN τ. LET US WRITE INSTANTON PATHS AS $\gamma^I(\tau)$, & DEFORMATION BY τ_1 AS $\gamma^I_{\tau_1}(\tau) = \gamma^I(\tau+\tau_1)$.

NEXT, WE APPEAL TO THE LOCALIZATION PRINCIPLE, WHICH DIRECTS OUR ATTENTION ONLY TO PATHS OF STEEPEST ASCENT. THEN, IN THE QUADRATIC APPROXIMATION, THE ACTION IS

$$S_E = \lambda[h(x_j) - h(x_i)] + \int \left(\tfrac{1}{2}|D_- \xi|^2 - D_- \bar\psi \cdot \psi\right) d\tau,$$

UNDER THE CHANGE OF VARIABLES $\phi^I = \gamma^I_{\tau_1} + \xi$.

THUS, WE HAVE THAT THERE IS ONE ξ ZERO MODE, ONE $\bar\psi$ ZERO MODE, & NO ψ ZERO MODES.

THE NON-ZERO MODE PART OF THE PATH-INTEGRAL IS

$$\frac{\det D_-}{\sqrt{\det D_-^\dagger D_-}} = \pm 1 .$$

THE ZERO-MODE INTEGRALS ARE THEN

$$\int_{-\infty}^{\infty} d\tau_1 \int d\bar{\psi}_0 \; \bar{\psi}^I \partial_I h \,\big|_{\tau=0} .$$

WE CAN WRITE $\bar{\psi}^I$ AS $\bar{\psi}_0 \dfrac{d\gamma^I_{\tau_1}}{d\tau} + \cdots$, WHERE THE DOTS ARE NON-ZERO MODE TERMS WHICH DO NOT CONTRIBUTE. THUS,

$$\begin{aligned} \bar{\psi}^I \partial_I h \,\big|_{\tau=0} &= \bar{\psi}_0 \frac{d\gamma^I_{\tau_1}}{d\tau} \partial_I h \,\big|_{\tau=0} \\ &= \bar{\psi}_0 \frac{d\gamma^I}{d\tau_1} \partial_I h(\gamma(\tau_1)), \end{aligned}$$

AT $\tau=0$

AND THEREFORE, $\displaystyle\int_{-\infty}^{\infty} d\tau_1 \frac{d\gamma^I}{d\tau_1}(\tau_1)\, \partial_I h(\gamma(\tau_1)) = h(x_j) - h(x_i).$

WHENCE $$\langle \Phi_j | Q \Phi_i \rangle = \sum_\gamma n_\gamma \, e^{-\lambda(h(x_j) - h(x_i))}$$

± 1

THE SIGN of n_γ IS FOUND AS follows;

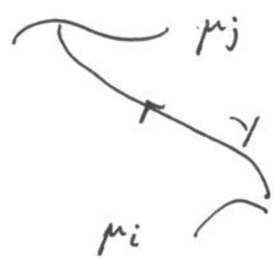
ALONG THE PATHS of STEEPEST ASCENT, n_γ IS ± 1

DEPENDING ON WHETHER THE ORIENTATION ~~of~~

DETERMINED BY $\bar{\Phi}_j \wedge * d_u \Phi_i$ MATCHES
THAT of $\mathcal{M}$ ALONG γ.

Φ_i DETERMINES A μ_i PLATE $T^{(-)}_{x_i} \mathcal{M}$ of
NEGATIVE EIGENMODES of THE HESSIAN AT x_i,
& THIS PLANE MAY BE TRANSPORTED ALONG γ.

THIS DETERMINES A SUBBUNDLE T_i^- of
$\gamma^* T\mathcal{M}$. SINCE $\mu_j = \mu_i + 1$, THERE IS ONE
MORE NEGATIVE EIGENVALUE AT x_j,
AND IT TURNS OUT THAT



T_i^- IS A SUBBUNDLE of T_j^-,
WHOSE COMPLIMENT IS SPANNED
BY v_γ, THE TANGENT VECTOR TO γ.

$Q\,\Phi_i$ DETERMINES AN ORIENTATION of $\mathbb{R} v_\gamma \oplus T_i^-$,
AND if IT MATCHES THE ORIENTATION ~~of~~ GIVEN
BY Φ_j, $n_\gamma = 1$; OTHERWISE $n_\gamma = -1$

VIA THE PATH INTEGRAL, WE FOUND THAT

$$Q\Psi_i = \sum_{j:\,\mu_j=\mu_i+1} \Psi_j \sum_{\gamma} n_\gamma\, e^{-\lambda[h(x_j)-h(x_i)]}.$$

NOW, LET US DEFINE A GRADED SPACE OF PERTURBATIVE GROUND STATES;

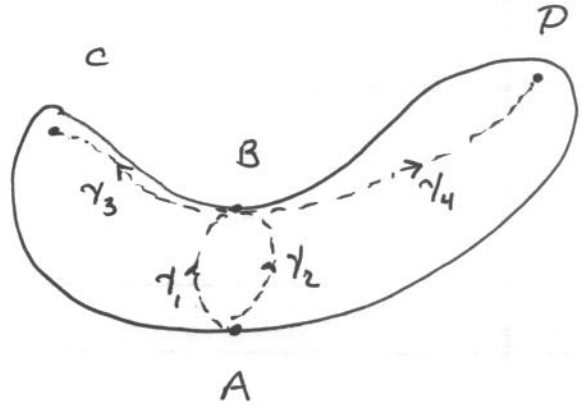
$$C^\mu = \bigoplus_{\mu_i=\mu} \mathbb{C}\,\Psi_i .$$

SINCE $Q^2=0$, & Q MAPS $\mu_i \to \mu_i+1$ Ψ_i's, WE HAVE THE MORSE-WITTEN COMPLEX

$$0 \longrightarrow C^0 \xrightarrow{Q} C^1 \xrightarrow{Q} \cdots \xrightarrow{Q} C^m \xrightarrow{Q} 0$$

THE SPACE OF SUSY GROUND STATES IS JUST THE COHOMOLOGY OF THIS COMPLEX.

CONSIDER



THERE ARE TWO INSTANTONS from A TO B, BUT THEIR ORIENTATIONS ARE opposite, SO

$$Q\,\Psi_A = 0.$$

THERE ARE TWO INSTANTONS WHICH ORIGINATE AT B. AGAIN, THEY ~~RO~~ ARE oppositely ORIENTED, So WE WILL HAVE

$$Q\,\Psi_B = \Psi_C - \Psi_D,$$

AND of COURSE $Q\,\Psi_D = Q\,\Psi_C = 0.$

So;
$$H^0(Q) = \mathbb{C}$$
$$H^1(Q) = 0 \qquad \text{(NO } \cancel{\text{EXACT}} \text{ CLOSED } 1\text{-forms)}$$
$$H^2(Q) = \mathbb{C}$$

$$\ker = \mathbb{C}\,\Psi_C \oplus \mathbb{C}\,\Psi_D$$

$$\mathrm{im} = \mathbb{C}(\Psi_C - \Psi_D)$$

$$\ker/\mathrm{im} \simeq \mathbb{C}$$