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Topology

The point of this talk is not to prove deep theorems, but to introduce the idea of systematic definition in mathematics.

~~The~~ This is the idea that given any statement, it can be reduced by definitions to ~~much of~~ a series of statements about axioms, ~~of an object.~~ which can be shown to be true or false by logic.

Of course in practice, ~~we~~ we use models in our minds ~~to~~ when working with objects, but when models fail, we can always fall back to the definitions.

①

CONSIDER A SET X.

A TOPOLOGY ON X IS A COLLECTION OF SUBSETS $\mathcal{T}$ SATISFYING THE FOLLOWING:

(1) ϕ & $X \in \mathcal{T}$

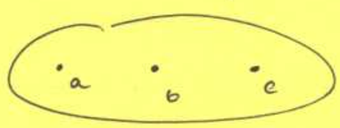
(2) THE UNION OF ANY ELEMENTS OF THE COLLECTION IS IN THE COLLECTION

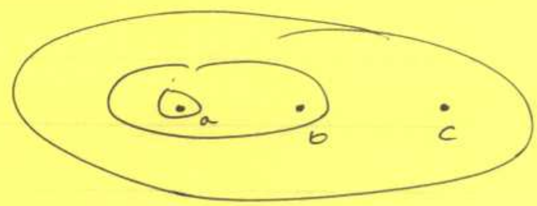
(3) THE INTERSECTION

A SET X ALONG WITH A TOPOLOGY $\mathcal{T}$ IS CALLED A TOPOLOGICAL SPACE. A SUBSET U OF X IS OPEN IFF $U \in \mathcal{T}$

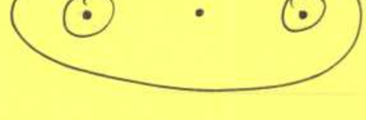
EX; $X = \{a, b, c\}$

$\mathcal{T} = \{\{a, b, c\}, \phi\}$ IS A TOPOLOGY (THE INDESCRETE, OR TRIVIAL)





$\{\{a\}, \{a, b\}, \{a, b, c\}, \phi\}$

NOTE THAT  IS NOT A TOPOLOGY

(2)

WE CAN COMPARE TWO TOPOLOGIES, BY CONTAINMENT

$$\text{if} \quad \mathcal{T}' \subset \mathcal{T}, \quad \text{THAT IS} \quad U \in \mathcal{T}' \Rightarrow U \in \mathcal{T},$$

THEN $\mathcal{T}$ IS <u>finer</u> THAN $\mathcal{T}'$

& $\mathcal{T}'$ IS <u>COARSER</u> THAN $\mathcal{T}$

Ex; $X = \{$ GEORGE, RINGO, PAUL, JOHN $\}$

$$\mathcal{T} = \left\{ \{G, R\}, \{P, J\}, \{G, R, P, J\}, \phi \right\}$$

Ex; $X = \mathbb{R}$

$$\mathcal{T} = \{ (-n, n),\ n \in \mathbb{Z} \}$$

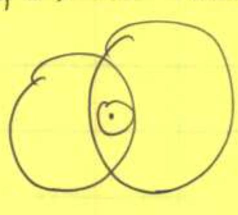
ANY POINT $r \in \mathbb{R}$, $\exists\ n \in \mathbb{Z}$ ST. $|r| < |n|$

A <u>BASIS</u> for A TOPOLOGY IS A COLLECTION $\mathcal{B}$ of SUBSETS of X (BASIS ELEMENTS) S.T.

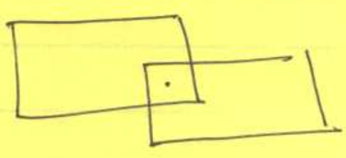
(1) $\forall\ x \in X$ $\exists$ AT LEAST ONE $B \in \mathcal{B} \ni x \in B$

(2) if $x \in B_1$ & $x \in B_2$, $\exists\ B_3 \subset B_1 \cap B_2$

Ex; (1) $\mathcal{B}$ = ALL INTERIORS of CIRCLES IN $\mathbb{R}^2$



(2) $\mathcal{B}$ = RECTANGULAR REGIONS IN $\mathbb{R}^2$



THE TOPOLOGY $\mathcal{T}$ GENERATED BY A BASIS IS THE COLL^n of ALL UNIONS & finite INTERSECTIONS of ELEMENTS of $\mathcal{B}$.

③

THE CARTESIAN PRODUCT of TWO SETS X & Y IS THE COLLECTION of PAIRS of ELEMENTS

$$X \times Y = \{ (x,y) \mid x \in X \text{ \& } y \in Y \}.$$

if $X, \mathcal{T}$ & $Y, \mathcal{S}$ ARE TOPO^l SPACES, THE PRODUCT TOPOLOGY ON $X \times Y$ IS

$$\{ U \times V \mid U \in \mathcal{T} \text{ \& } V \in \mathcal{S} \}$$

THRM: if $\mathcal{B}$ IS A BASIS for X & $\mathcal{C}$ for Y, $\mathcal{B} \times \mathcal{C}$ IS A BASIS for THE PRODUCT TOPOLOGY ON $X \times Y$.

Pf: WTS THAT (1) $\forall$ (x,y) $\exists$ $U \times V \in \mathcal{B} \times \mathcal{C}$ CONTAINING IT.

(2) if $(x,y) \in U_1 \times V_1, U_2 \times V_2$, $\exists$ $x \in U_3 \times V_3 \subset U_1 \times V_1 \cap U_2 \times V_2$

$$\mathcal{B} \times \mathcal{C} \equiv \{ U \times V \mid U \subset \mathcal{B} \text{ \& } V \in \mathcal{C} \}$$

TAKE $U \subset \mathcal{B}$ w/ $x \in U$ (SINCE $\mathcal{B}$ A BASIS)

& $V \subset \mathcal{C}$ w $y \in V$ (— " $\mathcal{C}$ — " —)

& $U \times V \ni (x,y)$. □

~~CLOSED~~

3a

THE SUBSPACE TOPOLOGY.

LET $X, \mathcal{T}$ BE A TOP^{l} SPACE, & S BE A SUBSET.

THEN, $\mathcal{T}_S \equiv \{ U \cap S, \ U \in \mathcal{T} \}$ forms A TOPOLOGY for S.

1. $\phi \in \mathcal{T}_S$; $\{ \phi \cap S = \phi \ \ \& \ \ \phi \in \mathcal{T} \}$
   $S \in \mathcal{T}_S$; $X \cap S = S \ \ \& \ \ X \in \mathcal{T}$
2. $U \cup V \in \mathcal{T}_S$ for $U, V \in \mathcal{T}_S$

3. $U = U' \cap S \qquad U' V' \in \mathcal{T}$

$$
\begin{aligned}
V &= V' \cap S \\
U \cap V &= (U' \cap S) \cap (V' \cap S) \\
&= \{ x \in U' \ \& \ x \in S \ \& \ x \in V' \ \& \ x \in S \} \\
&= \{ x \in U' \ \& \ x \in V' \ \& \ x \in S \} \\
&= (U' \cap V') \cap S
\end{aligned}
$$

$$
\begin{aligned}
U \cup V &= (x \in U' \ \& \ x \in S) \text{ or } \{ x \in V' \ \& \ x \in S \} \\
&= (x \in U' \text{ or } x \in V') \ \& \ x \in S \\
&= (U' \cup V') \cap S
\end{aligned}
$$

Closed Sets:

A subset $A \subset X$ is closed if $X - A$ is open.

Note ϕ, X are ALWAYS in a topology, that is by def^n they are open.

$$X - X = \phi$$
$$X - \phi = X$$

So X & ϕ are both open & closed!

How is a set diff^t than a door? A door must be open or closed, not both. A set may be open, closed both, or neither!

If X is a top^l space

(*)
(1) X & ϕ closed
(2) arbitrary intersections of cl. sets are closed
(3) finite unions of cl. sets are closed.

In fact this is an alternate def^n of topology. One can show that

(**)
(1) X & ϕ open
(2) arbitrary unions of opens are open
(3) finite intersections of ~~closed~~ opens are open

~~imply one another~~ $(**) \iff (*)$

(5)

A SPACE IS SAID TO BE HAUSDORFF IF $\forall$ DISTINCT x_1 & $x_2 \in X$, $\exists\ U_1, U_2 \subset X$; $x_1 \in U_1$ & $x_2 \in U_2$ S.T. $U_1 \cap U_2 = \emptyset$.

THRM; EVERY FINITE PT SET IN A HAUSDORFF SPACE IS CLOSED.
(EQUIV. TO 1-PT SETS ARE CLOSED)

CONSIDER $\{x_0\} \subset \mathcal{T}$.

$x_0 \in X$ & $x_0 \neq x \in X$

HAUS. $\Rightarrow \exists\ U\ V \ni V \cap U = \emptyset$

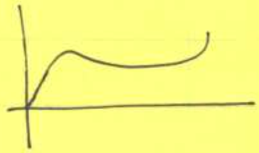
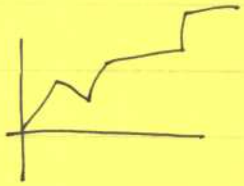
CONSIDER $\overline{\{x_0\}} = \{\cap \bar{U} \mid \bar{U}$ CLOSED & $x_0 \in \bar{U}\}$.

$X - U \ni x_0$ & $x \notin X - U$.

THUS $\overline{\{x_0\}}$ DOES NOT CONTAIN ANY OTHER POINT; $\overline{\{x_0\}} = \{x_0\}$. & x_0 IS CLOSED.

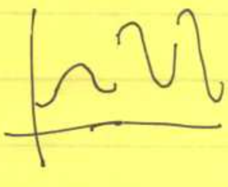
(6) Continuous function.

Intuitively, we understand the concept of a continuous map.

f  

$f: \mathbb{R} \to \mathbb{R}$

are continuous

(1)  ~~or not~~  are not

But why is this?

The top^l def^n: Let X & Y be top^l spaces. A function $f: X \to Y$ is continuous if ~~$f^{-1}(V)$ is open~~ . $V \subset Y$ open $\Rightarrow f^{-1}(V)$ open in X.

Here $f^{-1}(V) = \{ x \in X \mid f(x) \in V \}$.

What is the pre-image of the non-cts fcts?

(1) $\Rightarrow$ $f(x) = \lceil x \rceil$; the next highest integer

$f(1.2) = 2$

$f(1) = 1$

So; $f^{-1}(0,1) = 1$ a closed point.

(Equivalently preimage of cl. sets are closed)

Ex. CTS FCTS. $f: X \to Y$

(1) $f: X \to y \in Y$.

if $V \subset Y$ open, THEN $f^{-1}(V) = X$ or ϕ,

& (X & ϕ) ARE open.

(2) A IS A SUBSPACE of X

$i: A \hookrightarrow X$ IS CTS.

By Def^n of SUBSPACE TOPOLOGY.