Vector Fields on Manifolds

Last time, we discussed maps between manifolds, in particular, smooth maps. A smooth function f on a manifold M is a C^∞ map to the manifold $\mathbb{R}$ (or $\mathbb{C}$)

$$f: M \longrightarrow \mathbb{R}$$

Furthermore, if f & g are smooth functions, we can add them pointwise to obtain a new C^∞ function

$$(f+g)(p) \equiv f(p)+g(p)$$

Additionally, if $r \in \mathbb{R}$, we can write

$$(r \cdot f)(p) \equiv r \cdot (f(p)),$$

& we see that the space of smooth functions on M, denoted $C^\infty(M)$, is a vector space over $\mathbb{R}$ ($\mathbb{C}$). You can naïvely picture this space as $\mathbb{R}^\infty$. This is not the only structure; we can multiply, so $C^\infty(M)$ is also a ring. Now, a vector field is an object which acts on this vector space (technically, it is called an endomorphism), and satisfies some axioms.

The space of vector fields on M will be denoted

$$\mathfrak{X}(M)$$

(2) if $\underline{X} \in \mathcal{X}(M)$,

(1) $\underline{X}(f \cdot g) = \underline{X}(f) \cdot g + f \cdot \underline{X}(g) \quad \forall \quad f, g \in C^{\infty}(M)$

(2) $\underline{X}(r \cdot f) = r \cdot \underline{X}(f) \quad \forall \quad r \in \mathbb{R}$ & $f \in C^{\infty}(M)$

(3) $\underline{X}(f + g) = \underline{X}(f) + \underline{X}(g)$.

Apparently a vector field is a linear operator on the space of functions. These axioms also imply that $\mathcal{X}(M)$ itself is a vector space:

$$(\underline{X} + \underline{Y})(f \cdot g) = \underline{X}(f) \cdot g + f \cdot \underline{X}(g) + \underline{Y}(f) \cdot g + f \cdot \underline{Y}(g)$$

$$(r \cdot \underline{X})(f) = r \cdot \left(\underline{X}(f) \right)$$

Now, what do these vector fields look like on coördinate patches?

Remember that functions are maps from M to $\mathbb{R}$, so if $p \in M$ & $U \subset M$ is a coördinate neighborhood of p; $p \in U$, then locally on $\{(U, \varphi)\}$,

$$f(p) = f \circ \varphi \circ \varphi^{-1}(p) \sim f(x^i)$$

for coördinate functions $x^i : \varphi(U) \to \mathbb{R}$

③

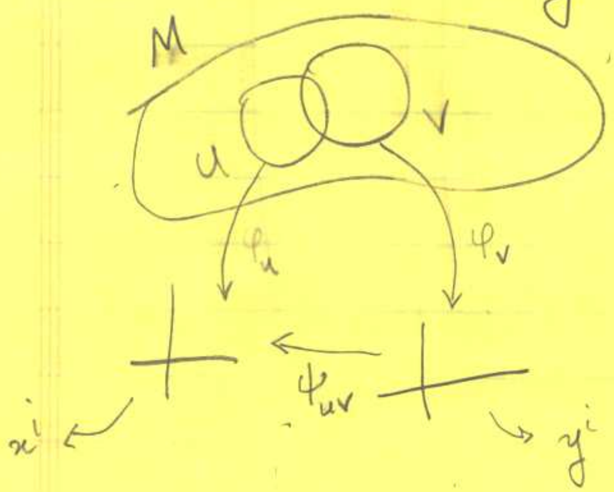
We act on f with a vector field $\underline{X}$ to obtain a new function $\underline{X}(f)$. For the coördinate functions, we define the new functions to be

$$\underline{X}(x^i) = X^i$$

so that $\underline{X}$ on $\varphi(U)$ is represented by the expression

$$\underline{X}_U = X^i \partial_{x^i}$$

Now, we must ask if this matches the definition in another coördinate system.



In the V coordinate patch, X becomes

$$\underline{X}_V = \tilde{X}^j \partial_{y^j}$$

We think of x^i as a function of y^i, so that

we must have $\underline{X}_U(x^i) = \underline{X}_V(x^i(y^j))$

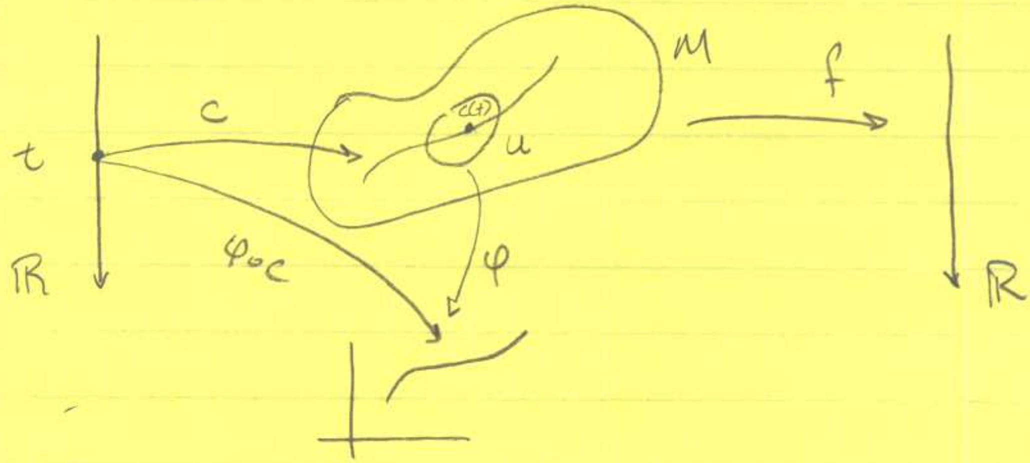
$$= \tilde{X}^j \frac{\partial x^i}{\partial y^j} = X^i$$

So that vector fields "transform like vectors"

④

VECTOR FIELDS ARE ALSO IMPORTANT WHEN DISCUSSING TANGENTS TO CURVES.

A CURVE IS A MAP $c : \mathbb{R} \to M$, USUALLY WRITTEN $c(t)$, WITH t A COÖRDINATE ON $\mathbb{R}$.



CONSIDER A FUNCTION $f : M \to \mathbb{R}$. A QUESTON TO ASK IS HOW DOES f CHANGE AS WE MOVE ALONG c. IF THIS WERE $\mathbb{R}^m$, WE KNOW THE ANSWER

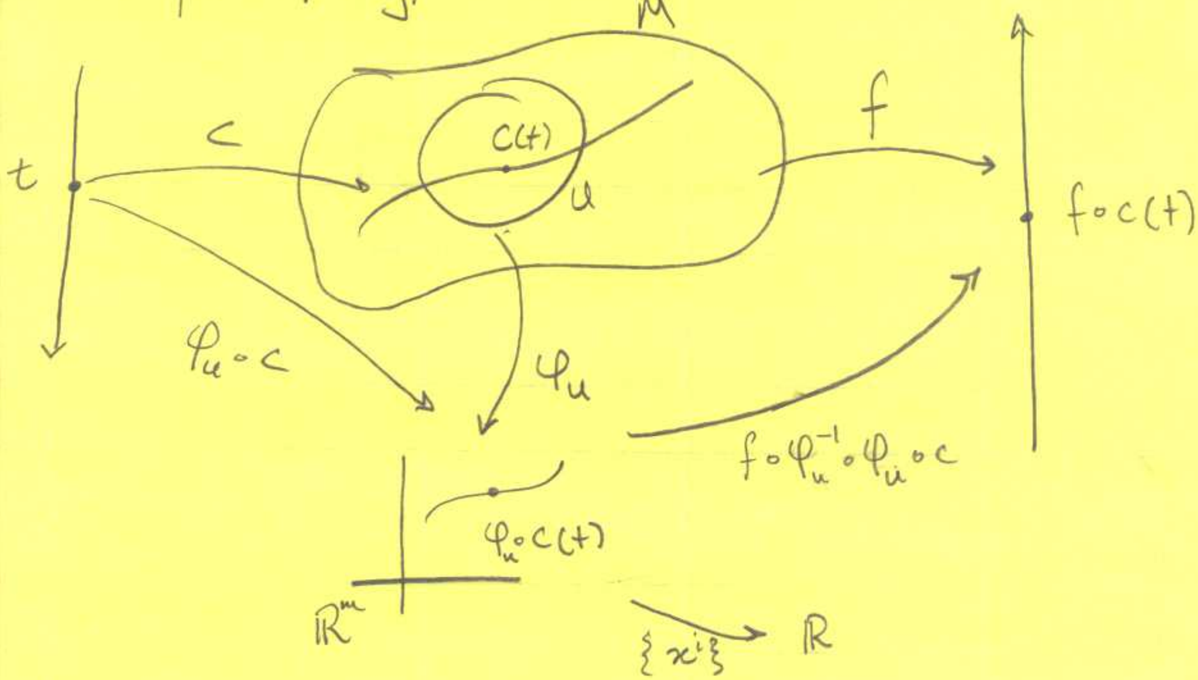
$$f(c(t)) - f(c(t_0)) \simeq \frac{\partial f}{\partial c^i} \frac{\partial c^i}{\partial t} (t - t_0)$$

BUT CONSIDER $f \circ c : R \to \mathbb{R}$, SO THE RATE OF CHANGE OF f ALONG c IS JUST

$$\frac{df(c(t))}{dt} = \frac{d(f \circ c)(t)}{dt}$$

THE DIFFICULTY LIES IN MAKING SENSE OF THE EXPRESSION LOCALLY...

IN ITS FULL GLORY;



SO ON $\varphi(U)$, WE HAVE $f\circ\varphi_u^{-1}\circ\varphi_u\circ c(t)$

INTRODUCING LOCAL COÖRDINATE FUNCTIONS x^i ON $\mathbb{R}^m$,

$$\underline{f\circ\varphi_u^{-1}\circ\{x^i\}^{-1}}\circ\{x^i\}\circ\varphi_u\circ c(t)$$

$$f_u(x^i(t))$$

& $$d_t f\circ c(t) = \frac{\partial f_u}{\partial x^i}\frac{\partial x^i}{\partial t}$$
$$= \left(\frac{\partial x^i}{\partial t}\right)\partial_{x^i} f_u$$

& IN ANOTHER COÖRDINATE SYSTEM, $x^i \rightsquigarrow x^i(y^j)$
$$= x^i\circ\varphi_{uv}\circ y^{-1}(y^j)$$

WITH THE LOCAL FUNCTION $f_v(y^j) = f\circ\varphi_v^{-1}\circ y^{-1}\circ y^j\circ\varphi_v\circ c(t)$

6

SINCE $\varphi_{uv} = \varphi_u \circ \varphi_v^{-1}$, $x^i(y^j) = x^i \circ \varphi_u \circ \varphi_v^{-1} \circ y^{-1}(y^j)$

SO THAT $f_v(y) = f \circ \varphi_v^{-1} \circ y^{-1} \circ \boxed{y^j \circ \varphi_v \circ c(t)}$

$$= \underbrace{f \circ \varphi_u^{-1} \circ x^{-1}}_{f_u\,(x^i(y^j))} \circ\, x^i(y^j)$$

& BY THE CHAIN RULE,

$$d_t\, f \circ c(t) = \frac{\partial f_v}{\partial y^j} \frac{\partial y^j}{\partial t} = \frac{\partial y^j}{\partial t} \frac{\partial x^i}{\partial y^j} \partial_{x^i} f$$

SO, IF WE IDENTIFY $\left.\frac{\partial x^i}{\partial t}\right|_{t=0} \equiv X^i$, WE CAN WRITE

$$\left.\frac{df(c(t))}{dt}\right|_{t=0} = X^i \partial_i f = X[f]$$

APPARENTLY, X^i IS THE TANGENT VECTOR TO $c(t)$ AT THE POINT $p = c(0)$. LETS CONSIDER $\underline{X}$ ACTING ON THE COÖRDINATE FUNCTIONS,

$$\underline{X}[x^i] = X^j \partial_j x^i$$
$$= X^i$$
$$= \frac{\partial x^i}{\partial t}$$

SO THE "TANGENT DIRECTION" TO A COÖRDINATE IS THE VELOCITY VECTOR, AS EXPECTED

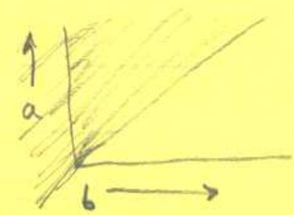
WITH THIS RELATIONSHIP, WE MAY CONSTRUCT A CLASS OF TANGENT VECTORS AT EACH POINT.

BEFORE DIVING INTO THIS, LET'S DEFINE AN EQUIVALENCE RELATION & AN EQUIVALENCE CLASS. IF S IS A SET, A RELATION R IS A SUBSET OF $S \times S$.

IF $(a, b) \in R \subset S^2$, WE WILL WRITE aRb.

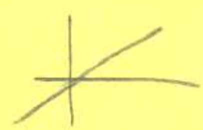
EXAMPLE; ① $(>) \subset \mathbb{R}^2$ IS A RELATION.

if $(a, b) \in (>)$, $a > b \Rightarrow$



② $= \subset \mathbb{R}^2$ IS A RELATION

if $(a, b) \in =$, $a = b \Rightarrow$



AN EQUIVALENCE RELATION $\sim$ IS A RELATION ON A SET S SATISFYING

(1) $a \sim a \quad \forall\ a \in S$

(2) if $a \sim b$, $b \sim a \quad \forall\ a, b \in S$

(3) if $a \sim b$ & $b \sim c$, $a \sim c \quad \forall\ a, b, c \in S$

EX; (1) $=$ IS AN EQUIV. RELT^n

(2) $>$ IS NOT; $a > b \Rightarrow b \not> a$

(3) $a \sim b$ if $a \bmod 2 = b \bmod 2$

EQUIVALENCE RELATIONS HAVE THE SPECIAL PROPERTY THAT THEY PARTITION THE SET INTO DISJOINT SUBSETS.

Define; $[a] = \{ s \in S \mid s \sim a \}$

NOTE, 1) $[a] \neq \phi$, SINCE $a \sim a$

2) if $a \not\sim b$, $[a] \cap [b] = \phi$

ASSUME IT'S NOT TRUE, THEN $\exists$ AN s WITH $s \sim a$ & $s \sim b$, BUT BY PROPERTY 2, THIS IMPLIES $a \sim b$, & $[a] = [b]$

THESE SUBSETS, $[a]$, ARE CALLED EQUIVALENCE CLASSES.

NOW, WE HAVE SEEN THAT THE TANGENT TO A CURVE AT A POINT IS A VECTOR FIELD ALONG THE CURVE. HOW DO WE DIFFERENTIATE TWO VECTORS, OR BETTER, HOW MANY TANGENT DIRECTIONS CAN THERE BE AT A POINT?

IT TURNS OUT THAT THE RIGHT WAY TO DEFINE THE TANGENT DIRECTIONS AS EQUIVALENCE CLASSES IN THE SET OF CURVES THROUGH A POINT

WE SAY $c_1(t) \sim c_2(t)$ if

(i) $C_1(0) = c_2(0) = p$

(ii) $d_t c_1(t)\big|_{t=0} = d_t c_2(t)\big|_{t=0}$ $\left(\text{"="} \ \frac{dx^i}{dt} \right)$

THEN A TANGENT VECTOR AT p IS $[c(t)]$ w/ $c(0) = p$