

1. SHEAVES

1.1. Presheaves. We begin with presheaves. A presheaf may be defined for any concrete category, examples of which include commutative rings, vector spaces, and abelian groups. A presheaf \mathcal{F} of a category (abelian groups) on a topological space X is an assignment of an abelian group (object of the category) $\mathcal{F}(U)$ to each open set $U \subset X$. In addition,

- (1) to each $V \subset U$, a morphism (group homomorphism) $r_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ called the restriction morphism, satisfying
- (2) $r_{UW} = r_{UV} \circ r_{VW}$ for each $W \subset V \subset U$, as well as $r_{UU} = \text{id}_{\mathcal{F}(U)}$ (0 for abelian groups).

Elements of $\mathcal{F}(U)$ are called sections. Unless explicitly noted, we will consider sheaves of abelian groups from now on.

Example 1.1.1. On a topological space X , we define $\mathbb{Z}(X)$, $\mathbb{R}(X)$, and $\mathbb{C}(X)$ to be constant \mathbb{Z} -, \mathbb{R} -, and \mathbb{C} -valued functions on X . On connected spaces, these presheaves are also sheaves (see example 1.2.1). \diamond

1.2. Sheaves. A sheaf on X is a presheaf \mathcal{F} on X satisfying two additional properties. Given a covering $\{U_i\}$ of $U \subset X$,

- (1) if σ is a section of $\mathcal{F}(U)$, and $\sigma|_{U_i} = 0 \forall i$, then $\sigma = 0$, and
- (2) (gluability) given $\sigma_i \in \mathcal{F}(U_i)$ satisfying

$$\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j} \quad \forall i, j,$$

then there exists a $\sigma \in \mathcal{F}(U)$ such that $\sigma|_{U_i} = \sigma_i \forall i$, and by (1), σ is unique.

Categorically speaking, we would say that given compatible elements σ_i (in the sense of (2)) on a cover, there exists a unique element of the sheaf $\mathcal{F}(U)$ which restricts to each of the given σ_i .

Example 1.2.1. For an easy example of a presheaf which is not necessarily a sheaf, let's consider the constant sheaf on a space X . Given an abelian group A , we set $\mathcal{A}(U) = A$ for all $U \subset X$ open. The restriction maps are simply the identity element of A . If X is a disconnected space with $X = U \cup V$, the presheaf $\mathcal{A}(X)$ is not a sheaf. Given any two distinct elements a, b of A , they may be taken to be sections $\mathcal{A}(U)$ and $\mathcal{A}(V)$ respectively. The sections satisfy $\sigma_U|_{U \cap V} = \sigma_V|_{U \cap V}$ trivially, but obviously do not 'glue' to a global section (a section of $\mathcal{F}(X)$). \diamond

Example 1.2.2. Commonly seen sheaves (from [GH78]).

Sheaves on a C^∞ manifold M :

- $C^\infty(U)$, C^∞ functions on U
- $C^*(U)$, nowhere vanishing C^∞ functions on U (a multiplicative group)
- $\mathfrak{A}^p(U)$, C^∞ p -forms on U
- $\mathfrak{Z}^p(U)$, closed C^∞ p -forms on U .

Sheaves on a complex manifold M :

- $\mathcal{O}(U)$, holomorphic functions on U
- $\mathcal{O}^*(U)$, nowhere vanishing holomorphic functions on U (a multiplicative group)
- $\Omega^p(U)$, holomorphic p -forms on U
- $\mathfrak{A}^{p,q}(U)$, C^∞ (p, q) -forms on U

- $\mathcal{Z}_{\bar{\partial}}^{p,q}(U)$, $\bar{\partial}$ -closed C^∞ forms of type (p, q) on U .
- $\mathcal{I}_V(U)$, holomorphic functions on U which vanish on $V \cap U$
- $\mathcal{O}(E)(U)$, holomorphic sections on U of a holomorphic vector bundle $E \xrightarrow{\pi} M$.

We may also notice that the sheaf \mathcal{O} is equivalent to sections of the trivial (complex) line bundle.

Example 1.2.3. Extension by zero. Given a sheaf $\mathcal{F}(N)$ on a subspace N of M , we may “extend \mathcal{F} by zero” to a sheaf $\tilde{\mathcal{F}}$ on M : for $U \subset M$ open, define

$$(1) \quad \tilde{\mathcal{F}}(U) = \mathcal{F}(U \cap N).$$

It is easy to check that $\tilde{\mathcal{F}}$ is a sheaf. ◇

Example 1.2.4. The Skyscraper sheaf. Let X be a space, with p a point in X . Take the sections as follows: If $p \in U \subset X$, then $\mathcal{S}(U) = \mathbb{C}$, and if $p \notin U$, $\mathcal{S}(U) = \{0\}$. It’s easy to see that this defines a sheaf, and that the support of the sheaf is the point p . ◇

Example 1.2.5. Direct Image Sheaf. Let $f : X \rightarrow Y$ be a continuous map of topological spaces. Then a sheaf \mathcal{F} on X induces a sheaf on Y by

$$(2) \quad (f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)),$$

where U is an open set in Y . ◇

Example 1.2.6. Subsheaves. A subsheaf of a sheaf \mathcal{F} is a sheaf \mathcal{F}' where on each open set U , $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$ and the restriction maps are inherited from \mathcal{F} . ◇

Example 1.2.7. The sheaf $\mathcal{O}(k)$. Consider the presheaf on \mathbb{P}^n defined by

$$(3) \quad \mathcal{O}(k)(U) = \{f \in \mathcal{O}(\pi^{-1}(U)) \mid f(\lambda x) = \lambda^k f(x) \forall \lambda \in \mathbb{C}^*\},$$

where $\pi : \mathbb{C} \rightarrow \mathbb{P}^n$ is the quotient map. The set of holomorphic homogeneous polynomials of degree k forms an additive abelian group, so we see that $\mathcal{O}(k)$ is a subsheaf of the direct image sheaf $\pi_*\mathcal{O}$. ◇

1.3. Sheaf and presheaf maps. For this section, take X to be a topological space, and $U \subset X$ open. A map of presheaves $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ on a space X is a collection of homomorphisms

$$(4) \quad \{\alpha_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)\}_{U \subset X},$$

such that

$$(5) \quad \begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\alpha_U} & \mathcal{G}(U) \\ r_{UV} \downarrow & & \downarrow r_{UV} \\ \mathcal{F}(V) & \xrightarrow{\alpha_V} & \mathcal{G}(V) \end{array}$$

commutes for all $V \subset U$ open. A map of sheaves is then a map of the underlying presheaves.

Example 1.3.1. Given a map of presheaves $\alpha: \mathcal{F} \rightarrow \mathcal{G}$, we define the following presheaves

$$\begin{aligned}\ker(\alpha)(U) &= \ker(\alpha_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)) \\ \text{coker}(\alpha)(U) &= \text{coker}(\alpha_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)) \\ \text{im}(\alpha)(U) &= \text{im}(\alpha_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U))\end{aligned}$$

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Proposition 1.1. *Given a map of sheaves $\alpha: \mathcal{F} \rightarrow \mathcal{G}$, $\text{Ker}(\alpha)$ is a sheaf.*

Proof. We check that the two extra properties of a sheaf are satisfied:

- (1) Let $\sigma \in \text{Ker}(\alpha)(U) \subset \mathcal{F}(U)$, and $\underline{U} = \{U_i\}$ be a covering of U . Then $\sigma|_{U_i} = 0$ implies that $\sigma = 0$ since \mathcal{F} is a sheaf.
- (2) Assume that $\tau_i \in \text{Ker}(\alpha)(U_i)$ agree on overlaps. Since α is a sheaf map, we have $\alpha(\tau_i) = \alpha(\tau)|_{U_i}$, and since \mathcal{G} is a sheaf, we can glue; $\alpha(\tau) = 0$, so that $\tau \in \text{Ker}(\alpha)(U)$. □

While the kernel of a sheaf map always satisfies the properties of a sheaf, the cokernel and image presheaves may not. Given sheaves \mathbb{Z} and C^∞ , for $U \subset X$ we have the inclusion homomorphism $\mathbb{Z}(U) \xrightarrow{i_U} C^\infty(U)$, and the exponential map $C^\infty(U) \xrightarrow{e^{2\pi i}} C^*(U)$. Take $X = S^1$ and cover it with open sets $U = (-\epsilon, \pi + \epsilon)$ and $V = (\pi - \epsilon, 2\pi + \epsilon)$.

Proposition 1.2. *The image presheaf $U \mapsto (\text{im } e^{2\pi i})(U)$ is not a sheaf.*

Proof. Naïvely, this sheaf should be all of C^* , but as we shall see, sections in the image may not glue properly. The exponential map on each open set gives a map of presheaves

$$(6) \quad C^\infty(U) \xrightarrow{e^{2\pi i}} C^*(U).$$

The problem comes from trying to interpret this as a map of sheaves. We examine sections of the sheaf C^∞ :

$$\begin{aligned}f_U : \theta &\longmapsto \theta & -\epsilon < \theta < \pi + \epsilon \\ f_V : \theta &\longmapsto \theta & \pi - \epsilon < \theta < 2\pi + \epsilon.\end{aligned}$$

with $f_U|_{U \cap V} \neq f_V|_{U \cap V}$

Clearly we have that

$$(7) \quad p(f_U)|_{U \cap V} = p(f_V)|_{U \cap V},$$

which implies that we should be able to glue to a global section of the image presheaf. It is also clear that no such section exists, so the presheaf $\text{im } e^{2\pi i}$ is not a sheaf. □

1.4. Sheafification. The problem in the above example was that the open sets we chose were “too large”. To remedy this, we define something “very local”.

Definition 1.4.1. The stalk \mathcal{F}_p of a presheaf \mathcal{F} at a point p is defined to be

$$(8) \quad \mathcal{F}_p \equiv \frac{\{(U, s) | U \text{ open and } p \in U, s \in \mathcal{F}(U)\}}{(U, s) \sim (V, t)},$$

where $(U, s) \sim (V, t)$ if there exists a $W \subset U \cap V$ open such that $s|_W = t|_W$. If our presheaf is \mathcal{O} , the stalk is simply germs of holomorphic functions at a point.

The stalk at a point is an abelian group, with addition defined as follows:

$$(9) \quad (U, f) + (V, g) = (U \cap V, f|_{U \cap V} + g|_{U \cap V}).$$

Given some $\sigma \in \mathcal{F}(U)$, we write σ_p for the image of σ in \mathcal{F}_p .

Example 1.4.1. Given a subsheaf \mathcal{F}' of \mathcal{F} , at any point p , the stalk \mathcal{F}'_p is a subgroup of the stalk \mathcal{F}_p . \diamond

Example 1.4.2. Given a skyscraper sheaf with support at the point p , the stalk of the sheaf at p is \mathbb{C} , while the stalk at any other point is $\{0\}$. \diamond

Now we are ready to define the sheafification \mathcal{F}^+ of a presheaf \mathcal{F} .

Definition 1.4.2. The sheafification of \mathcal{F} on an open set U is the set of all functions $s: U \rightarrow \bigcup_{p \in U} \mathcal{F}_p$ satisfying

- (1) $s(p) \in \mathcal{F}_p$
- (2) for all $p \in U$, there exists an open neighbourhood $W \subset U$ of p and a section $t \in \mathcal{F}(W)$ such that $s(q) = t|_q$ for all $q \in W$.

The process of sheafification forces you to not consider large open sets, but to examine germs, which are very local, and match them around the set. We can now define the sheaves $\text{Im}(\alpha)$ and $\text{Coker}(\alpha)$; they are the sheafifications of the presheaves $\text{im}(\alpha)$ and $\text{coker}(\alpha)$, respectively.

1.5. Exact Sequences of Sheaves. A map of sheaves $\phi: \mathcal{F} \rightarrow \mathcal{G}$ is injective if $\text{Ker}(\phi) = 0$. Note that ϕ is injective iff the maps $\phi(U): \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ are injective for all $U \subset X$ open. The sheaf $\text{Im}(\phi)$ is defined to be the sheaf associated with the presheaf image of ϕ . $\text{Im}(\phi)$ may be identified with a subsheaf of \mathcal{G} , and so we define ϕ to be surjective if $\text{Im}(\phi) = \mathcal{G}$. Recall that a complex (of sheaves) is a collection of sheaves \mathcal{C}_n and maps $\phi_n: \mathcal{C}_n \rightarrow \mathcal{C}_{n+1}$, usually written

$$(10) \quad \dots \xrightarrow{\phi_{n-2}} \mathcal{C}_{n-1} \xrightarrow{\phi_{n-1}} \mathcal{C}_n \xrightarrow{\phi_n} \mathcal{C}_{n+1} \xrightarrow{\phi_{n+1}} \dots,$$

with the property that $\text{ker}(\phi_n) \subset \text{im}(\phi_{n-1})$. This is a fancy way of saying that $\phi_n \circ \phi_{n-1} = 0$ for all n . An exact sequence is a complex where $\text{ker}(\phi_n) = \text{im}(\phi_{n-1})$.

Definition 1.5.1. A map of (pre)sheaves α is an isomorphism of (pre)sheaves if each homomorphism α_U is an isomorphism.

Example 1.5.1. Let 0 denote the constant sheaf of the abelian group $\{0\}$. If we have a short exact sequence of sheaves $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$, then $\mathcal{F} \simeq \mathcal{G}$. \diamond

Example 1.5.2. If we have a short exact sequence of sheaves $0 \rightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \rightarrow 0$, then $\mathcal{G} \simeq \text{Coker}(\alpha) = \mathcal{F}/\mathcal{E}$. \diamond

2. ČECH AND SHEAF COHOMOLOGY

2.1. Cochains and the coboundary operator. We begin by defining cochain groups:

$$(11) \quad C^p(\underline{U}, \mathcal{F}) = \bigoplus \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_p})$$

An element $\sigma \in C^p(\underline{U}, \mathcal{O})$ is a collection of sections on $p + 1$ intersections, and is called a p -cochain.

Example 2.1.1. For the sheaf of homomorphic functions, \mathcal{F} , the first few cochain groups take the form

$$(12) \quad \begin{aligned} C^0(\underline{\mathcal{U}}, \mathcal{O}) &= \{f_\alpha \in \mathcal{O}(U_\alpha)\} \\ C^1(\underline{\mathcal{U}}, \mathcal{O}) &= \{h_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)\} \\ C^2(\underline{\mathcal{U}}, \mathcal{O}) &= \{g_{\alpha\beta\gamma} \in \mathcal{O}(U_\alpha \cap U_\beta \cap U_\gamma)\}. \end{aligned}$$

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Now, we define a coboundary operator

$$(13) \quad \delta: C^p(\underline{\mathcal{U}}, \mathcal{F}) \rightarrow C^{p+1}(\underline{\mathcal{U}}, \mathcal{F})$$

by the formula (for $\sigma \in C^p(\underline{\mathcal{U}}, \mathcal{F})$)

$$(14) \quad (\delta\sigma)_{i_0, \dots, i_{p+1}} = \sum_{\ell=0}^{p+1} (-1)^\ell \sigma_{i_0, \dots, \hat{i}_\ell, \dots, i_{p+1}} \Big|_{U_{i_0} \cap \dots \cap U_{i_{p+1}}}.$$

In words: the image of σ under δ on the intersection $U_{i_0, \dots, i_{p+1}} = U_{i_0} \cap \dots \cap U_{i_{p+1}}$ is the alternating sum of σ on $U_{i_0, \dots, \hat{i}_\ell, \dots, i_{p+1}}$, where the \hat{i}_ℓ denotes U_{i_ℓ} is excluded from the intersection. Cochains σ which satisfy $\delta\sigma = 0$ are called cocycles, and if $\sigma = \delta\tau$ for some τ , σ is called a coboundary. One may check that $\delta^2 = 0 \forall p$.

Example 2.1.2. Let $\underline{\mathcal{U}} = \{U_\alpha, U_\beta, U_\gamma\}$, with $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Then, for $\sigma \in C^1(\underline{\mathcal{U}}, \mathcal{F})$, we have

$$(15) \quad \begin{aligned} (\delta\sigma)_{\alpha\beta} &= \sigma_\alpha - \sigma_\beta \equiv \tau_{\alpha\beta} \\ &\text{and} \\ (\delta\tau)_{\alpha\beta\gamma} &= \tau_{\beta\gamma} - \tau_{\alpha\gamma} - \tau_{\alpha\beta} \\ &= \sigma_\beta - \sigma_\gamma - \sigma_\alpha + \sigma_\gamma + \sigma_\alpha - \sigma_\beta = 0 \end{aligned}$$

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If we define $Z^p(\underline{\mathcal{U}}, \mathcal{F}) = \text{Ker}\delta \subset C^p(\underline{\mathcal{U}}, \mathcal{F})$, we will have the p^{th} Čech cohomology group

$$(16) \quad \check{H}^p(\underline{\mathcal{U}}, \mathcal{F}) = \frac{Z^p(\underline{\mathcal{U}}, \mathcal{F})}{\delta C^{p-1}(\underline{\mathcal{U}}, \mathcal{F})}.$$

Let us examine properties of the Čech cohomology groups. We note that since there are no “-1” forms, for any cover $\underline{\mathcal{U}}$ of M , $\check{H}^0(\underline{\mathcal{U}}, \mathcal{F})$ is composed of those cochains which match on all intersections of the cover; that is, global sections of the sheaf. Thus we have that

$$(17) \quad \check{H}^0(\underline{\mathcal{U}}, \mathcal{F}) = \mathcal{F}(M).$$

True sheaf cohomology is defined in terms of derived functors. However, in some cases sheaf and Čech cohomologies coincide.

Theorem 2.1. Let $H^0(\underline{\mathcal{U}}, \mathcal{F})$ denote the 0^{th} sheaf cohomology. Then, $\check{H}^0(\underline{\mathcal{U}}, \mathcal{F}) \simeq H^0(\underline{\mathcal{U}}, \mathcal{F})$.

Theorem 2.2. (Leray) Given a covering $\underline{\mathcal{U}}$, if $H^i(V, \mathcal{F}) = 0$ for any finite intersection $V = U_{i_1} \cap \dots \cap U_{i_n}$, then for all $i \geq 0$

$$(18) \quad \check{H}^i(\underline{\mathcal{U}}, \mathcal{F}) \simeq H^i(X, \mathcal{F}).$$

2.3. Vanishing theorems. Vanishing theorems, coupled with long exact cohomology sequences, give us extremely powerful techniques for computing cohomology groups.

Theorem 2.4. ($\bar{\partial}$ -Poincaré Lemma) *If $\Delta(r)$ is a polycylinder (a complex ball) of radius r in \mathbb{C}^n , $H_{\bar{\partial}}^{p,q}(\Delta) = 0$ for $q \geq 1$. Note that r may be taken to ∞ , so that $\Delta = \mathbb{C}^n$.*

Proof. See [GH78], §0.2. □

Definition 2.3.1. A sheaf \mathcal{S} is **fine** if for every locally finite covering of the space by sets $\{U_\alpha\}$, there exists a collection of endomorphisms $\{\phi_\alpha\}$ of the sheaf so that $\phi_\alpha s_x = 0$ if $x \notin U_\alpha$, and so that $\sum_\alpha \phi_\alpha = id_{\mathcal{S}}$.

What this means is that we can find a partition of unity subordinate to some locally finite cover: that is, for $\underline{U} = \{U_\alpha\}$, maps $\eta_\alpha: \mathcal{F}(U_\alpha) \rightarrow \mathcal{F}(U)$ such that the support of $(\eta_\alpha \sigma)$ is contained in U_α and $\sum \eta_\alpha(\sigma|_{U_\alpha}) = \sigma \in \mathcal{F}(U)$.

Theorem 2.5. *If \mathcal{F} is fine, $H^p(M, \mathcal{F}) = 0$ for $p > 0$.*

Proof. See [GH78], §0.3. □

2.4. Čech -de Rham isomorphism. By the ordinary Poincaré theorem, the long sheaf sequence

$$(25) \quad 0 \rightarrow \mathbb{R} \xrightarrow{i} \mathfrak{A}^0 \xrightarrow{d} \mathfrak{A}^1 \xrightarrow{d} \mathfrak{A}^2 \rightarrow \dots$$

is exact. Thus, we will have short exact sequences

$$(26) \quad 0 \rightarrow \mathbb{R} \xrightarrow{i} \mathfrak{A}^0 \xrightarrow{d} \mathcal{Z}^1 \rightarrow 0$$

and

$$(27) \quad 0 \rightarrow \mathcal{Z}^p \xrightarrow{i} \mathfrak{A}^p \xrightarrow{d} \mathcal{Z}^{p+1} \rightarrow 0$$

for all p . Now, (26) induces the long exact sequence

$$(28) \quad 0 \rightarrow \dots \rightarrow H^q(M, \mathfrak{A}^p) \rightarrow H^q(M, \mathcal{Z}^{p+1}) \rightarrow H^{q+1}(M, \mathcal{Z}^p) \rightarrow H^{q+1}(M, \mathfrak{A}^p) \rightarrow \dots$$

and since \mathfrak{A}^p is a fine sheaf, $H^q(M, \mathfrak{A}^p) = 0$ for all $q > 0, p \geq 0$. Thus,

$$(29) \quad H^q(M, \mathcal{Z}^{p+1}) \simeq H^{q+1}(M, \mathcal{Z}^p).$$

Similarly, the exact sequence (27) induces the long exact sequence

$$(30) \quad 0 \rightarrow \dots \rightarrow H^q(M, \mathfrak{A}^0) \rightarrow H^q(M, \mathcal{Z}^1) \rightarrow H^{q+1}(M, \mathbb{R}) \rightarrow H^{q+1}(M, \mathfrak{A}^0) \rightarrow \dots$$

so that $H^q(M, \mathcal{Z}^1) \simeq H^{q+1}(M, \mathbb{R})$ for all $q > 0$. Thus, we have that

$$\begin{aligned}
(31) \quad \check{H}^p(M, \mathbb{R}) &\cong H^{p-1}(M, \mathcal{L}^1) \\
&\cong H^{p-2}(M, \mathcal{L}^2) \\
&\quad \vdots \\
&\cong H^1(M, \mathcal{L}^{p-1}) \\
&\cong \frac{H^0(M, \mathcal{L}^p)}{\delta H^0(M, \mathcal{A}^{p-1})} \\
&= \frac{Z^p(M)}{dA^{p-1}(M)} \\
&\equiv H_{\text{DR}}^p(M).
\end{aligned}$$

Similarly, we have the *Dolbeault Theorem*:

$$(32) \quad H^q(M, \Omega^p) \cong H_{\bar{\partial}}^{p,q}(M).$$

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