Last time, we assumed that we could compute the spectrum of our theory exactly. Unfortunately, this is not always possible. However, an approximation method known as perturbation theory can give an exact answer for the Witten Index.
Consider a single variable potential theory, with the lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}[h'(x)]^2 + \frac{i}{2}[\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi] - h''(x)\bar{\psi}\psi, \tag{1}$$

where $\frac{1}{2}[h'(x)]^2$ is the potential, $V(x)$. This lagrangian is supersymmetric under the transformations

$$\begin{aligned} \delta x &= \epsilon\bar{\psi} - \bar{\epsilon}\psi && (2)\\ \delta\psi &= \epsilon[i\dot{x} + h'(x)] && (3)\\ \delta\bar{\psi} &= \bar{\epsilon}[-i\dot{x} + h'(x)]. && (4) \end{aligned}$$

The Nöther charges associated with this symmetry are

$$Q = \bar{\psi}[i\dot{x} + h'(x)] \qquad \bar{Q} = \psi[-i\dot{x} + h'(x)] \tag{5}$$

Canonical quantization $\big([x,p] = i, \{\psi, \bar{\psi}\} = 1\big)$ yields the hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}[h']^2 + \frac{1}{2}h''[\bar{\psi}, \psi]. \tag{6}$$

It is a fact that this theory's Hilbert space is

$$\mathscr{H} = L^2(\mathbb{R}, \mathbb{C})\,|0\rangle \oplus L^2(\mathbb{R}, \mathbb{C})\bar{\psi}\,|0\rangle\,. \tag{7}$$

In the basis $(|0\rangle\,, \bar{\psi}\,|0\rangle)$, the Nöther charges take the form

$$Q = \bar{\psi}[ip + h'(x)] = \begin{pmatrix} 0 & 0 \\ \frac{d}{dx} + h'(x) & 0 \end{pmatrix} \tag{8}$$

$$Q = \psi[-ip + h'(x)] = \begin{pmatrix} 0 & -\frac{d}{dx} + h'(x) \\ 0 & 0 \end{pmatrix}. \tag{9}$$

We want to find ground states... i.e. a $|\psi\rangle = f_1(x)\,|0\rangle + f_2(x)\bar{\psi}\,|0\rangle$ such that $Q\,|\psi\rangle = \bar{Q}\,|\psi\rangle = 0$. These conditions are equivalent to the differential equations

$$\begin{aligned} \left(\frac{d}{dx} + h'(x)\right) f_1(x) &= 0 \\ \left(-\frac{d}{dx} + h'(x)\right) f_2(x) &= 0, \end{aligned} \tag{10}$$

which have the solutions

$$\begin{aligned} f_1(x) &= c_1 e^{-h(x)} \\ f_2(x) &= c_2 e^{h(x)}. \end{aligned} \tag{11}$$

There is a constraint on f_1 and f_2; they must be square integrable. Assuming polynomial growth of $|h(x)|$ as $x \to \pm\infty$, there are three possibilities.
For a specific example, consider the supersymmetric simple harmonic oscillator,

$$\begin{aligned} h(x) &= \frac{1}{2}\omega x^2 \\ V(x) &= \frac{1}{2}\omega^2 x^2 \end{aligned} \tag{12}$$

$$\text{if } \omega > 0 \qquad \Psi = e^{-\frac{\omega}{2}x^2}|0\rangle\,, \quad \mathrm{Tr}(-1)^F = 1$$
$$\text{if } \omega < 0 \qquad \Psi = e^{\frac{\omega}{2}x^2}\bar{\psi}|0\rangle\,, \quad \mathrm{Tr}(-1)^F = -1 \tag{13}$$

Now that we have found that this theory admits one supersymmetric ground state, and we have calculated the Witten Index exactly, let us apply perturbative techniques and compare answers.
First, we deform the theory by rescaling the function h; $h \to \lambda h, \quad \lambda \gg 1$. Under this deformation, the Hamiltonian becomes

$$H = \frac{1}{2}p^2 + \frac{1}{2}\lambda^2[h'(x)]^2 + \frac{1}{2}\lambda[\bar{\psi}, \psi]. \tag{14}$$

In the limit $\lambda \to \infty$, the wavefunctions of the lowest energy states become shareply peaked around the lowest values of $[h'(x)]^2$. Now, suppose that $h(x)$ has a critical point at $x = x_i$. Near x_i, we can write $h(x)$ as

$$h(x) = h(x_i) + \frac{1}{2}h''(x_i)(x - x_i)^2 + \cdots . \tag{15}$$

If we rescale $x - x_i$ to $\widetilde{x} - \widetilde{x}_i = \sqrt{\lambda}(x - x_i)$, the Hamiltonian organizes into a $\lambda^{-1/2}$ perturbation series.

$$H = \lambda\left\{\frac{1}{2}\widetilde{p}^2 + \frac{1}{2}[h''(x_i)]^2(\widetilde{x} - \widetilde{x}_i)^2 + \frac{1}{2}h''(x_i)[\bar{\psi}, \psi]\right\} + \lambda^{1/2}(\cdots) + (\cdots) + \mathcal{O}(\lambda^{-1/2}) \tag{16}$$

So, the leading order term in the Hamiltonian is

$$H_0 = \frac{1}{2}p^2 + \frac{1}{2}\lambda^2[h''(x_i)]^2(x - x_i)^2 + \frac{1}{2}\lambda h''(x_i)[\bar{\psi}, \psi], \tag{17}$$

which is just the supersymmetric super-harmonic oscillator Hamiltonian, with $\omega = \lambda h''(x_i)$. Thus, the perturbative ground state wavefunction around the critical point x_i is

$$\Psi_i = e^{-\frac{1}{2}\lambda h''(x_i)(x-x_i)^2}|0\rangle + \cdots \qquad \text{if } h''(x_i) > 0$$
$$\Psi_i = e^{\frac{1}{2}\lambda h''(x_i)(x-x_i)^2}\bar{\psi}|0\rangle + \cdots \qquad \text{if } h''(x_i) < 0, \tag{18}$$

where $(\cdots)$ means additions of $\mathcal{O}(\lambda^{-1/2}$ due to $\mathcal{O}(\lambda^{1/2}$ terms in the Hamiltonian. With these additions, the energy is zero to each order, and the state is thus a ground state to each order. The Witten Index of this state is then

$$1 \text{ if } h''(x_i) > 0$$
$$-1 \text{ if } h''(x_i) < 0. \tag{19}$$

If there are N critical points of h, then there are N perturbative ground states, $\Psi_1, \cdots, \Psi_N$. If we consider the sum of the perturbative series around each point as a deformation of the actual theory, we can compute $\mathrm{Tr}(-1)^F$;

$$\mathrm{Tr}(-1)^F = \sum_{i=1}^{N} \mathrm{sign}[h''(x_i)] \tag{20}$$

So, the perturbation theory gives the correct Witten Index. Unfortunately it tells us that there are N ground states. Furthermore, the Ψ_i are ground states to all orders in perturbation theory: There is a non-perturbative effect at work, which gives energy to most of these states. This effect is known as "Quantum Tunnelling". This will be discussed in the next talk.
Now, consider a multi-variable potential theory, with m bosonic and $2m$ fermionic terms; x^I and $\psi^I, \bar{\psi}^I$, and Hamiltonian

$$H = \frac{1}{2}\sum_I \left\{p_I^2 + [\partial_I h(x)]^2\right\} + \frac{1}{2}\partial_I\partial_J h(x)[\bar{\psi}^I, \psi^I]. \tag{21}$$

Assume that h has isolated and non-degenerate critical points (a Morse function). We again rescale $h \to \lambda h$, and choose coordinates $\xi^I_{(i)}$ to diagonalize the hessian around each critical point x_i.

$$h(x) = h(x_i) + \sum_I c_I^{(i)} (\xi^I_{(i)})^2 + \cdots\ , \tag{22}$$

where $c_I^{(i)}$ are the components of the diagonalized hessian. In the large λ limit, the ground state wavefunctions are localizzed near critical points, and we have the approximate wavefunctions

$$\Psi_i = \exp\left\{-\sum_{I=1}^{n} \lambda |c_I^{(i)}| (\xi^I_{(i)})^2\right\} \prod_{J: C_J^{(i)}<0} \bar{\psi}^J \left|0\right\rangle . \tag{23}$$

The number of $\bar{\psi}^I$ is given by the number of negative eigenvalues of the hessian at x_i. This is just the Morse Index μ_i. Thus,

$$\mathrm{Tr}(-1)^F = \sum_i (-1)^{\mu_i}. \tag{24}$$

Now, we shall consider a different method for calculating the Witten Index. Consider the theory on a circle of circumerence β. It turns out that the partition function is

$$\mathrm{Tr}e^{-\beta H} = \int DXD\psi D\bar{\psi}|_{\mathrm{AP}} e^{S_E(X,\psi,\bar{\psi})}, \tag{25}$$

where AP signifies anti-periodic boundary conditions on the fermions. Anti-periodic boundary conditions are necessary because if we insert two fermions, say at t_1 and t_2, then taking $t_1 \to t_1 + \beta$ moves ψ past $\bar{\psi}$, contributing a $-$ sign (fermions must come in pairs due to fermion number symmetry). If there is a $(-1)^F$ insertion, taking $t_1 \to t_1 + \beta$ would pick up two $-$ signs. Thus, we see that the Witten Index is

$$\mathrm{Tr}(-1)^F e^{\beta H} = \int DXD\psi D\bar{\psi}|_P e^{S_E(X,\psi,\bar{\psi})}. \tag{26}$$

Now, let us calculate $\mathrm{Tr}(-1)^F$ for the single variable potential theory. The Euclidean action for this theory is

$$S_E = \int_0^\beta d\tau \left\{\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + \frac{1}{2}[h'(x)]^2 + \bar{\psi}\frac{d}{d\tau}\psi + h''(x)\bar{\psi}\psi\right\}. \tag{27}$$

The supersymmetry transformations are

$$\delta x = \epsilon\bar{\psi} - \bar{\epsilon}\psi \tag{28}$$

$$\delta\psi = \epsilon\left(-\frac{dx}{d\tau} + h'(x)\right) \tag{29}$$

$$\delta\bar{\psi} = \bar{\epsilon}\left(\frac{dx}{d\tau} + h'(x)\right). \tag{30}$$

The localization principle tells us that the path-integral recieves contributions from fermionic fixed points of the supersymmetry transformations;

$$\begin{aligned}\delta\psi &= \delta\bar{\psi} = 0 \\ \frac{dx}{d\tau} &= h'(x) = 0.\end{aligned} \tag{31}$$

This means that the path-integral is concentrated around constant maps to the critical points of h. If we set $\xi = x - x_i$, the action in the quadratic approximation is

$$S_E^{(i)} = \int_0^\beta \left\{ \frac{1}{2}\xi \left(-\frac{d^2}{d\tau^2} + h''(x_i)^2 \right) \xi + \bar{\psi} \left(\frac{d}{d\tau} + h''(x_i) \right) \psi \right\}. \tag{32}$$

Thus, the i^{th} critical point's contribution to the Witten Index is

$$\begin{aligned}\mathrm{Tr}(-1)^F_{(i)} &= \int DXD\psi D\bar{\psi}|_P e^{-S_E^(i)} \\ &= \frac{\det\left(\frac{d}{d\tau} + h'(x_i)\right)}{\sqrt{\det\left(\frac{d^2}{d\tau^2} + h''(x_i)^2\right)}} \\ &= \frac{h''(x_i)}{|h''(x_i)|} \\ &= \mathrm{sign}[h''(x_i)].\end{aligned} \tag{33}$$

Thus, the Witten Index is

$$\mathrm{Tr}(-1)^F = \sum_i \mathrm{sign}[h''(x_i)]. \tag{34}$$