Math 116 | Sample Exam #1 | October, 2008.

Instructions: This exam consists of five problems. Do all five, showing your work and explaining your assertions. Give yourself 50 minutes. Each problem is worth 20 points.

1. Suppose that $T \subset S \subset \mathbb{R}$, and that $T \neq \varnothing$.
   a) Show that if $S$ has a supremum, then so does $T$.
   b) Determine whether the converse also holds.

2. Let $f(x) = x^2[x]$, where $[x]$ is the greatest integer in $x$.
   a) On which closed intervals is $f$ integrable?
   b) Evaluate $\int_0^2 f(x)dx$ if it exists.

3. Let $f$ be a function with the property that $-x^2 \leq f(x) \leq x^2$ for all $x$.
   a) Determine whether $f$ is continuous at $x = 0$.
   b) Determine whether $f$ is differentiable at $x = 0$. If it is, find $f'(0)$.

4. a) In terms of $\varepsilon, \delta$, write out explicitly the meaning of the assertion that the function $f(x) = 3x$ is continuous at $x = 0$.
   b) Explicitly prove this assertion by finding an appropriate $\delta$ for each $\varepsilon$.

5. Prove or disprove each of the following assertions:
   a) For every $v \in \mathbb{R}^2$ there exists a non-zero vector $w \in \mathbb{R}^2$ such that $v \perp w$ (i.e. $v \cdot w = 0$).
   b) If $v \in \mathbb{R}^2$ has the property that $v \cdot w \geq 0$ for all $w \in \mathbb{R}^2$, then $v = 0$.