Chap 8: Inner products

$F = \mathbb{R}, \mathbb{C}$

On $\mathbb{R}^n$: dot product

– standard inner product

$v = (a_1, \dots, a_n),\ w = (b_1, \dots, b_n)$

$v \cdot w = \sum a_i b_i \in \mathbb{R}$

Geometrically:

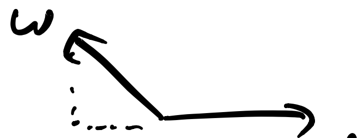
v, w $(\neq 0)$: lie in a plane.

in this configuration: projection of w onto v

$v \cdot w = (\text{length } v)(\text{length } \mathrm{pr}_v w)$

$= (\text{length } v)(\text{length } w)\cos\theta$

$= (\text{length } w)(\text{length } \mathrm{pr}_w v)$



use signed length of proj. in general.

Length in terms of dot product.

v, length $\|v\| \geq 0$ $\quad \|v\|^2 = v \cdot v = \sum a_i^2$

$$\|v\| = \sqrt{\underbrace{\sum a_i^2}_{\geq 0}} \geq 0 \; ; \quad = 0 \Leftrightarrow v = 0$$

$V = \mathbb{R}^n$

$V \times V \longrightarrow \mathbb{R} \qquad (v, w) \longmapsto v \cdot w$

Properties:

- real valued
- linear as function in 1st vbl:

  $(v_1 + v_2) \cdot w = v_1 \cdot w + v_2 \cdot w$

  $(cv) \cdot w = c\,(v \cdot w)$
- symmetric:

  $v \cdot w = w \cdot v$

  linear as fn of 2nd vbl
- $\therefore$ <u>bilinear</u>

- positive definite

  $\underbrace{v \cdot v \geq 0}_{\text{if just have this: pos. semi-def}}$, & > 0 unless $v = 0$.

In gen'l, if we have v.s. V over $\mathbb{R}$,

& $V \times V \to \mathbb{R}$

called an <u>inner product</u> if it has these properties.

Ex 2 p271 H+K $\quad \mathbb{R}^2$

$(v|w)$

$v = (a_1, a_2) \quad w = (b_1, b_2)$

$$(v|w) = a_1 b_1 - \underset{-1}{a_2 b_1} - \underset{-1}{a_1 b_2} + 4a_2 b_2$$

Symmetric

bilinear

pos def – need to check

$$(v|v) = a_1^2 - 2a_1 a_2 + 4a_2^2$$

$$= \underset{\geq 0}{(a_1 - a_2)^2} + \underset{\geq 0}{3a_2^2}$$

$= 0 \iff a_1 = a_2 = 0$ ✓.

<u>This is an inner product.</u>

On $\mathbb{C}$: $\quad V$ v.s. / $\mathbb{C}$

Variant

Cx dot product on $\mathbb{C}^n$

standard hermitian inner product

$v = (a_1, \_\_, a_n) \in \mathbb{C}^n$

$w = (b_1, \_\_, b_n) \in \mathbb{C}^n$

Define $v \cdot w = \sum a_i \overline{b_i}$

E.g. If $a_i \in \mathbb{R}$, & $b_i \in \mathbb{R}$

$v \in \mathbb{R}^n \subset \mathbb{C}^n$, $w \in \mathbb{R}^n \subset \mathbb{C}^n$

$v \cdot w = \sum a_i \overline{b_i} = \sum a_i b_i = v \cdot w$

in cx sense

in real sense.

Properties of cx dot products

- cx valued $V \times V \to \mathbb{C}$

$(V = \mathbb{C}^n)$

- linear in 1st vbl

$v \cdot w = \sum a_i \overline{b_i}$

$w \cdot v = \sum b_i \overline{a_i}$

$b_i \overline{a_i} = \overline{a_i \overline{b_i}}$

$= \overline{a_i} \, \overline{\overline{b_i}}$

$w \cdot v = \overline{v \cdot w}$

- Conjugate symmetric

— ? in 2° vbl

$V \cdot (w_1 + w_2) = V \cdot w_1 + V \cdot w_2$

$V \cdot (c w) = \bar{c}\,(v \cdot w)$

Conjugate linear in 2° vbl.

$V \times V \to \mathbb{C}$

Sesquilinear ($1\frac{1}{2}$ linear)

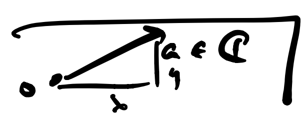
Pos def?

$V = (a_1, \ldots, a_n) \in \mathbb{C}^n \qquad a_j \in \mathbb{C}$

$V \cdot V = a_1 \bar{a}_1 + \cdots + a_n \bar{a}_n$

$a \in \mathbb{C} \qquad a = x + iy \qquad x, y \in \mathbb{R}$

$a\bar{a} = (x+iy)(x-iy) = x^2 + y^2 \geq 0 \leftarrow |a|^2$



$= 0 \iff x, y = 0$

$\iff a = 0$

$|a| = \sqrt{x^2 + y^2}$

$\in \mathbb{R}$

$V \cdot V = a_1 \bar{a}_1 + \cdots + a_n \bar{a}_n = |a_1|^2 + \cdots + |a_n|^2 \geq 0$

& $= 0 \iff$ all $a_i = 0 \iff V = 0$.

Norm: $\|v\| \geq 0$, $\|v\|^2 = V \cdot V \in \mathbb{R}$, $v \cdot v \geq 0$.

$v \in \mathbb{C}^n \rightsquigarrow \tilde{v} \in \mathbb{R}^{2n}$

$\|$

$(a_1, \dots, a_n) \rightsquigarrow (x_1, y_1, x_2, y_2, \dots, x_n, y_n)$

$a_j = x_j + i y_j$

$$\underbrace{\|v\|}_{\text{in } \mathbb{C}^n} = \sqrt{|a_1|^2 + \dots + |a_n|^2}$$

$$= \sqrt{x_1^2 + y_1^2 + \dots + x_n^2 + y_n^2}$$

$$= \underbrace{\|\tilde{v}\|}_{\text{in } \mathbb{R}^{2n}}$$

Norm agrees: $\mathbb{C}^n$ vs. $\mathbb{R}^{2n}$.

Caution: $v, w \in \mathbb{C}^n$

$\rightsquigarrow \tilde{v}, \tilde{w} \in \mathbb{R}^{2n}$

Usually, $\underbrace{v \cdot w}_{\text{in } \mathbb{C}^n} \neq \underbrace{\tilde{v} \cdot \tilde{w}}_{\text{in } \mathbb{R}^{2n}}$

usually $\notin \mathbb{R}$. $\quad \in \mathbb{R}$

In gen'l, if have $\underbrace{V}_{\text{vs}/\mathbb{C}} \times V \to \mathbb{C}$

satisfying above properties

(cx val, sesquilin, pos def):

(conj sym, lin in 1st var)

cx (hermitian) inner product on V.

Ex. of inf dim'l vs with an inner product. (Impt in analysis)

$/\mathbb{R}$, $V = \{\text{cont. fns}: [0,1] \to \mathbb{R}\}$.

Inner prod: $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$

$/\mathbb{C}$ $V = \{\text{cont fns}: [0,1] \to \mathbb{C}\}$.

inner prod: $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$

V f.d.v.s $/\mathbb{R}$ or $\mathbb{C}$.

Basis $\mathcal{B}$: $v_1, \dots, v_n$.

We can write any vector in terms of $\mathcal{B}$

$v = a_1v_1 + \dots + a_nv_n$ $\quad$ a_i's: coords of v wrt $\mathcal{B}$.

$v = \sum a_iv_i$

$w = \sum b_iv_i$

Define $\langle v, w\rangle = \sum_{i=1}^n a_ib_i$ if $/\mathbb{R}$

or $\sum_{i=1}^n a_i\overline{b_i}$ if $/\mathbb{C}$.

Generalizes dot prod: Can $V = \mathbb{R}^n$ or $\mathbb{C}^n$ & $\mathcal{B}$: $e_1, \dots, e_n$.

V vs $/\mathbb{R}$ or $\mathbb{C}$

$\langle \cdot, \cdot \rangle$ inner product

$\rightsquigarrow \| \cdot \|$ norm.

$\|v\|^2 = \langle v, v \rangle$.

Conversely From $\|\cdot\|$,

we can get $\langle \cdot, \cdot \rangle$!

Polarization identities $/\mathbb{R}$, & $/\mathbb{C}$:

$/\mathbb{R}$ (8-3)

$$\langle v, w \rangle = \frac{1}{4}\|v+w\|^2 - \frac{1}{4}\|v-w\|^2$$

p^{274}

$/\mathbb{C}$ (8-4)

$$\langle v, w \rangle = \frac{1}{4}\|v+w\|^2 + \frac{i}{4}\|v+iw\|^2 - \frac{1}{4}\|v-w\|^2 - \frac{i}{4}\|v-iw\|^2$$

Can verify by writing out each

$$\|z\|^2 = \langle z, z \rangle$$

& use bilinearity (or sesquilinearity) & expand.

$\therefore$ to know $\langle \cdot, \cdot \rangle$, it suffices to know $\|\cdot\|$.

To know $\langle v, w \rangle$ for all $v, w \in V$,

suffices to know this on the basis vectors $v_1, \dots, v_n$

$$a_{ij} = \langle v_i, v_j \rangle \in \mathbb{R}, \text{ or } \mathbb{C} \quad (\text{if } / \mathbb{C})$$

Recall: $v = \sum_{i=1}^{n} b_i v_i$

$w = \sum_{j=1}^{n} c_j v_j$

$/\mathbb{R}$

(*) $\underline{\underline{\langle v, w \rangle}} = \sum_{i,j=1}^{n} b_i c_j \langle v_i, v_j \rangle = \underline{\sum_{i,j} b_i c_j a_{ij}}$

$\bar{c}_j$ if $/ \mathbb{C}$.

$A = (a_{ij})$ $n \times n$

mx of the inner product $\langle \cdot, \cdot \rangle$ wrt this basis

Ex. $\mathbb{R}^n$, std basis, dot product

$$a_{ij} = \langle e_i, e_j \rangle = \delta_{ij}$$

$\therefore A = I$

Ex. $\mathbb{R}^2$, std basis,

inner prod: $v=(a_1,a_2)$, $w=(b_1,b_2)$

$$\langle v,w\rangle = \underset{1}{a_1 b_1} - \underset{-1}{a_2 b_1} - \underset{-1}{a_1 b_2} + \underset{4}{4 a_2 b_2}$$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$$

In gen'l, /$\mathbb{R}$: A is symmetric.

$$\begin{array}{ccc} a_{ij} & = & a_{ji} \\ \| & & \| \\ \langle v_i, v_j\rangle & = & \langle v_j, v_i\rangle \end{array}$$

$$\langle v,w\rangle = \underset{1\times n}{v}\,\underset{n\times n}{A}\,\underset{n\times 1}{[w]} \in \mathbb{R} \text{ by } (*).$$

/$\mathbb{C}$: $a_{ji} = \bar{a}_{ij}$, Conjugate sym. mx.

"Hermitian mx"

$$A^t = \bar{A}$$

$$\langle v,w\rangle = vA[\bar{w}]$$

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§8.2 V vs /$\mathbb{R}$ or $\mathbb{C}$,

together w inn. prod: inner product space

$\langle \cdot, \cdot \rangle \rightsquigarrow \|\cdot\|$ norm.

$\left(\begin{array}{c} \langle \cdot, \cdot \rangle \text{ is dot prod} \\ \Downarrow \\ \|\cdot\| : \text{usu length} \end{array} \right)$

$$\|v\| = \sqrt{v \cdot v}$$

Properties of norm: $\|\cdot\| : V \to \mathbb{R}$

1. $\|c\alpha\| = |c| \, \|\alpha\|$
2. $\|\alpha\| \geq 0$, & $= 0 \iff \alpha = 0$
3. $|\langle \alpha, \beta \rangle| \leq \|\alpha\| \, \|\beta\|$

   Cauchy-Schwarz inequality
4. $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$

   $\triangle$-inequality

Follow from def of $\|\cdot\|$ in terms of $\langle \cdot, \cdot \rangle$ & properties of inner prod.

(Th 1, p277, H&K). ①, ②, ④

Caution: Not every norm comes from an inner product.

Ex. $\mathbb{R}^n$, $v = (a_1 \ldots a_n)$

$$\|v\| = \sum_{i=1}^{n} |a_i|. \qquad \ell_1\text{-norm.}$$

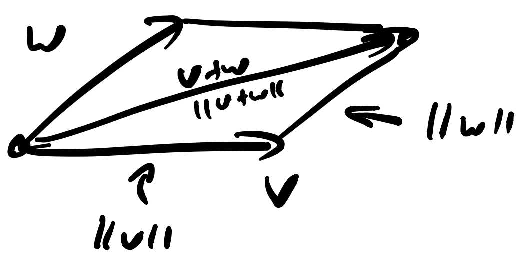
For dot products & assoc. $\|\cdot\|$:

C-S: $|\alpha\cdot\beta| = \|\alpha\|\,\|\beta\|\,\underbrace{|\cos\theta|}_{\le 1}$

/$\mathbb{R}$

$$\le \|\alpha\|\,\|\beta\|.$$

Δ-ineq:



Ex. of C-S:

dot prod on $V=\mathbb{R}^n$ $\quad v=(a_1,\dots,a_n)$, $w=(b_1,\dots,b_n)$

$$\left|\sum_{i=1}^{n} a_i b_i\right| \le \left|\sum a_i^2\right|^{1/2} \left|\sum b_i^2\right|^{1/2}$$

for all $a_1,\dots,a_n, b_1,\dots,b_n \in \mathbb{R}$.

Ex. of C-S:

$V=\{\text{cont fns}: [0,1]\to\mathbb{R}\}$.

$$\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$$

$$\left|\int_0^1 f(x)g(x)\,dx\right| \le \left|\int_0^1 f(x)^2\,dx\right|^{1/2} \left|\int_0^1 g(x)^2\,dx\right|^{1/2}$$

In $\mathbb{R}^n$, with dot product:

If $v, w \neq 0$, making an $\measuredangle$ of Θ,

$$v \perp w \iff \Theta = \pm 90^\circ \iff \cos\theta = 0$$
$$\iff v \cdot w = 0$$

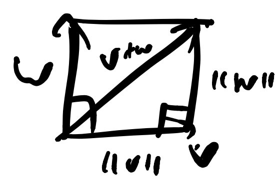
View $0 \perp v$, for all v.

More generally, in any inner prod. sp., define $v \perp w$ if $\langle v, w \rangle = 0$

v, w are orthogonal

As in $\mathbb{R}^n$, we have

$$v \perp w \iff \|v+w\|^2 = \|v\|^2 + \|w\|^2$$



Pythagorean thm

for <u>all</u> inner product spaces

(by bilinearity)

Say $S \subset V$ subset

inn. pr. sp.

Let $S^\perp = \{ v \in V \mid \forall w \in S,\ v \perp w \}$

Orthogonal complement ("S perp")

Thm: $S^\perp$ is a subspace of V

(easy to check)

Ex. $V=\mathbb{R}^3$, $S=\{e_1\}$ dot product

$S^\perp$ = span of $\{e_2, e_3\}$ $W = \operatorname{span} S$

= y-z plane. = x-axis

$W^\perp = S^\perp$

Ex. $V=\mathbb{R}^2$, $S=\{(1,2)\}$ $W = \operatorname{span} S$

$(-2,1) \in S^\perp$ $W^\perp = S^\perp$

$S^\perp$ = span of $\{(-2,1)\}$.

In general, given $S \subset V$,

let $W = \operatorname{span} S$.

Get $S^\perp = W^\perp$

Ex. $V^\perp = 0$

(b/c if $v \in V^\perp$ then

$\langle v,v\rangle = 0$, so $v = 0$. ✓)

Ex. $0 = \{0\}$. $0^\perp = V$

$S \subset V$ subset

Say S is <u>orthogonal</u> if $\mathbb{R}^2$, dot. pr.

any two vectors in S are orthog. Ex. $S=\{(1,2),(-2,1)\}$

Say S is orthonormal if
orthogonal & all the vectors in S
have unit length ("normalized")

Ex. $\{e_1, \ldots, e_k\} \subset \mathbb{R}^n$ $\quad k \leq n$
orthonormal

Ex $V = \{$ cont fns $[0,1] \to \mathbb{R}\}$
Infinite set
$\{1, \cos 2\pi x, \sin 2\pi x, \cos 4\pi x, \sin 4\pi x, \ldots\}$
is orthogonal.

Thm (H+K, Th 2, p 278)
An orthogonal set of non-0 vectors
is lin. independent.

Pf. $S = \{v_1 \ldots v_n\}$. $v_i \neq 0$. orthog.
If $\sum_{i=1}^{n} a_i v_i = 0$, WTS all $a_j = 0$.

$$0 = \langle 0, v_j \rangle = \langle \sum_{i=1}^{n} a_i v_i , v_j \rangle$$

$$= \sum_{i=1}^{n} a_i \langle v_i , v_j \rangle \underset{\text{orthog}}{=} a_j \|v_j\|^2$$

$a_j \|v_j\|^2 = 0$ $\quad v_j \neq 0$, $\therefore \|v_j\|^2 \neq 0$.
$\therefore a_j = 0.$

$S = \{v_1 \dots v_k\} \subset V$, orthog set of non-0 vectors

$\therefore$ lin ind.

If $v \in \operatorname{span} S$ then $v = \sum_{i=1}^{k} \underbrace{a_i}_{\text{scalars}} v_i$,

<u>uniquely</u>. (lin ind)

There is a <u>formula</u> for a_i's:

$$\boxed{a_i = \frac{\langle v, v_i \rangle}{\|v_i\|^2}}$$

Pf. $\langle v, v_i \rangle = \langle \sum_j a_j v_j, v_i \rangle$

$= \sum_j a_j \langle v_j, v_i \rangle \underset{\text{orthog}}{=} a_i \|v_i\|^2$ ✓

In particular, if the orthog. set $v_1 - v_n$ spans V, then it's a basis, & we get a formula for coords of any vector. $v = \sum a_i v_i$

Even better: if have o.n.b. then $\|v_i\|^2 = 1$, so

$a_i = \langle v, v_i \rangle$. (Simpler)

How to construct an orthogonal basis?

Gram-Schmidt orthogonalization

— uses above formula

Given $\beta_1, \ldots, \beta_n$ lin ind
in an inner prod. space,
We can find $\alpha_1, \ldots \alpha_n$, orthogonal
(+ lin. ind) with the <u>same</u>
span as $\beta_1 \ldots \beta_n$.

In partic, if βs are a basis for V,
then α's are an orthog basis for V.

Then: $\frac{\alpha_1}{\|\alpha_1\|}, \ldots \frac{\alpha_n}{\|\alpha_n\|}$ is o.n.b. for V.

Procedure: Start with $\beta_1, \ldots, \beta_n$. ← lin ind.

Construct $\alpha_1, \ldots, \alpha_n$ inductively

$\alpha_1 = \beta_1$.

$$\alpha_2 = \beta_2 - \frac{\langle \beta_2, \alpha_1 \rangle}{\|\alpha_1\|^2} \alpha_1$$

$$\alpha_3 = \beta_3 - \frac{\langle \beta_3, \alpha_1 \rangle}{\|\alpha_1\|^2}\alpha_1 - \frac{\langle \beta_3, \alpha_2 \rangle}{\|\alpha_2\|^2}\alpha_2$$

etc.

Easy to check that each α_i is $\perp$ to the prev ones,

+ α's have same span as β's.

(see H + K, pp 280 - 281.

see Example 12, p282)

Ex. Can apply to get an orthog. basis of some given subspace of $\mathbb{R}^n$ (w.r.t dot product)

e.g. plane in $\mathbb{R}^3$ (or $\mathbb{C}^3$)

- See prob 6 on Supplementary Study problems for Exam 3

- see #4 on PS14.

Once we have an orthog. basis,

$\rightsquigarrow$ get o.n.b. by normalization.

Can also use G-S to extend a given orthog. set of non-0 vectors to an orthog. basis:

Orthog set $\alpha_1, \ldots \alpha_k$ $(\neq 0)$

Extend to a basis: add $\beta_{k+1}, \ldots \beta_n$

Apply G-S to $\alpha_1, \ldots \alpha_k, \beta_{k+1}, \ldots \beta_n$

1st k steps do nothing

remaining steps replace $\beta_{k+1}, \ldots \beta_n$ by $\alpha_{k+1}, \ldots \alpha_n$

st $\{\alpha_1, \ldots \alpha_k, \ldots \alpha_n\}$ is orthog, same span, so basis. orthogonal.

V vs with $\langle , \rangle$.

$\text{span}\{\alpha_1, \ldots, \alpha_k\} = W$

Full orthog basis $\underbrace{\alpha_1, \ldots \alpha_k}_{\text{span}=W}, \underbrace{\alpha_{k+1}, \ldots, \alpha_n}_{\text{span}=W'}$

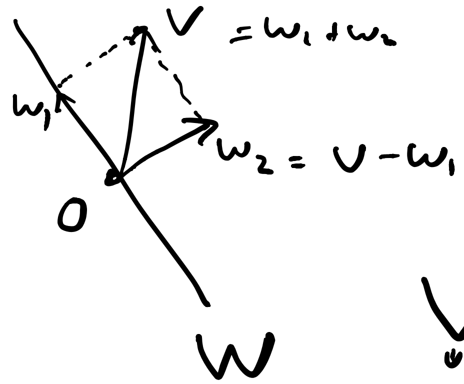
$W' = W^\perp$

$V = W \oplus W^\perp$ b/c $W \cap W^\perp = 0$

Sometimes written $W \oplus W^{\perp}$

- orthogonal direct sum.

V



$w_1 \perp w_2$

$w_1 \in W$

$w_2 \in W^{\perp}$

$$\begin{array}{ccc} V & \longrightarrow & W \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v & \longmapsto & w_1 \end{array}$$

lin tr.

Orthogonal projection

$\mathrm{im} = W \qquad \ker = W^{\perp}$

w_1 is the point on W that is closest to $v \in V$

"best approx to v on W"

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§8.3 Adjoints

V fd inn prod space $/ F = \mathbb{R}$ or $\mathbb{C}$

$\dim = n$.

Take $\beta \in V$

$$\begin{array}{ccc} \alpha & \longmapsto & \langle \alpha, \beta \rangle \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f_\beta : V & \longrightarrow & F \end{array}$$

lin. map.

$f_\beta \in V^*$, dual space

linear functional on V

Take o.n.b. of V: $v_1, \dots v_n$

$$\beta = \sum_{i=1}^{n} b_i v_i \qquad b_i = \langle \beta, v_i \rangle.$$

$$f_\beta(\alpha) = \langle \alpha, \beta \rangle = \langle \alpha, \sum b_i v_i \rangle$$

$$= \sum b_i \underbrace{\langle \alpha, v_i \rangle}_{a_i} = \sum a_i b_i \quad \text{if } /\mathbb{R}$$

$\alpha = \sum a_i v_i$

$$\text{or} = \sum \bar{b}_i \langle \alpha, v_i \rangle = \sum a_i \bar{b}_i \quad \text{if } / \mathbb{C}.$$

Mx of $f_\beta : \underset{n}{V} \longrightarrow \underset{1}{F}$

$1 \times n$ is $(b_1, \dots, b_n)$ $/\mathbb{R}$

$(\bar{b}_1, \dots, \bar{b}_n)$ $/\mathbb{C}$

Conversely every lin fnl $f \in V^*$

$(f : V \longrightarrow F)$ is this form f_β

for some β.

$\beta \longleftrightarrow f$ bijection

$$\begin{array}{ccc} \beta & \longleftrightarrow & f \\ \cap & & \cap \\ V & \xrightarrow[(\beta \mapsto f_\beta)]{} & V^* \end{array}$$

$\Phi: V \xrightarrow{\text{bij}} V^*$ $\quad \beta \mapsto f_\beta$

easy to check: Φ is a lin. transf

$\therefore$ isomorphism.

$\langle\cdot,\cdot\rangle \rightsquigarrow \Phi$ (vs. $V \xrightarrow{\sim} V^{**}$ even w/o $\langle\cdot,\cdot\rangle$)

<u>Adjoint</u> of a lin transf:

Thm (H + K, Th 7, p 293 + Cor, p 294)

Let $T: V \to V$ lin op.

on a fin dim inn. pr. sp

w o.n.b. $v_1, \_ v_n$ (wrt $\langle\cdot,\cdot\rangle$).

Let A be the mx of T (wrt this basis)

Let $A^* = \begin{cases} A^t & \text{if } /\mathbb{R} \\ \bar{A}^t & \text{if } /\mathbb{C} \end{cases}$

Let $T^*: V \to V$ be the

corresp lin tr. (wrt this basis)

Then $\forall \alpha, \beta \in V$:

(*) $\langle T\alpha, \beta\rangle = \langle \alpha, T^*\beta\rangle$

(*): Adjoint property.

Pf is by direct computation.

T^* is the <u>adjoint</u> of T.

(this is uniquely determined by (*)).

Q: What is the relationship between above $T^*: V \to V$ (adjoint) to the dual $T^t: V^* \to V^*$? also called T^*

$T \longleftrightarrow A$ wrt basis of V

$T^* \longleftrightarrow A^*$ --- ($= A^t$ if $/\mathbb{R}$)

$T^t \longleftrightarrow A^t$ --- dual basis of V^*

If $/\mathbb{R}$, $T^*: V \to V$

$T^t: V^* \to V^*$

Both have mx A^t

$\langle , \rangle \leadsto \Phi: V \xrightarrow{\sim} V^*$

If identify V, V^* via Φ: T^t becomes T^*.

$V, \langle , \rangle$

$T: V \to V$ $\qquad$ T lin op.

$\updownarrow$ $\qquad$ $T \in \operatorname{End} V$

A, mx wrt o.n.b. $\qquad$ $\operatorname{Hom}(V,V)$

$\rightsquigarrow$ adjoint $T^*: V \to V$

$T^* \in \operatorname{End} V$

$$a: \operatorname{End} V \longrightarrow \operatorname{End} V$$
$$T \longmapsto T^*$$

corresp

$$M_n(F) \longrightarrow M_n(F)$$
$$A \longmapsto A^* = \begin{cases} A^t & \text{if } /\mathbb{R} \\ \bar{A}^t & \text{if } /\mathbb{C} \end{cases}$$

If $/\mathbb{R}$, lin transf $A \mapsto A^t$

If $/\mathbb{C}$, conj linear $A \mapsto \bar{A}^t$

$(TU)^* = U^* T^*$ $\qquad$ since true for transpose $/\mathbb{R}$

Similarly $/\mathbb{C}$, with conj

$T^{**} = T$ $\qquad$ b/c ok for mx's.

If $T^* = T$ we say
T is <u>self-adjoint</u>.

Which lin transf's are self-adjoint?

$T \longleftrightarrow A$ wrt onb under $\langle , \rangle$

$T = T^* \iff A = A^*$

If $/\mathbb{R}$: $A = A^t$: A is symmetric
$a_{ji} = a_{ij}$

If $/\mathbb{C}$, $A = \bar{A}^t$, $A^t = \bar{A}$
A is conj sym (Hermitian)
$a_{ji} = \overline{a_{ij}}$

V inn. pr. sp $\langle \cdot, \cdot \rangle$

lin op $T: V \to V$, iso of v.s.'s
ignoring $\langle , \rangle$.

Under what circumstances does T preserve $\langle , \rangle$? i.e. $\langle v, w \rangle = \langle Tv, Tw \rangle$
— <u>iso of inner product spaces</u>

T preserves inner product $\langle\, ; \rangle$

$\Longleftrightarrow T$ preserves norm $\qquad \|\cdot\|$

$\|T(v)\| = \|v\|$. (PS13 #5)

Q: Which T preserve norm?

Ans Nec & Suff condition:

$$\boxed{T^* = T^{-1}}$$

Reason: $\langle v, v' \rangle \overset{?}{=} \langle Tv, Tv' \rangle \quad \forall v, v'$

$?=$

$= \langle v, T^*Tv' \rangle \quad$ (Adj property)

$\forall v, v'$

$\langle v, v' \rangle = \langle v, T^*Tv' \rangle \quad \forall v, v'$

$\langle v, T^*Tv' - v' \rangle = 0 \Leftarrow v \perp \underbrace{T^*Tv' - v'}_{\in V^\perp = 0}$

$T^*Tv' - v' = 0$. $\quad T^*Tv' = v' \quad \forall v'$

$\Longleftrightarrow \underline{T^*T = I} \Longleftrightarrow T^* = T^{-1}$. —

Impt class of lin op's T:

pres. inn prod; pres. norm; $T^* = T^{-1}$.

These are called <u>unitary</u>.

Reason for terminology:

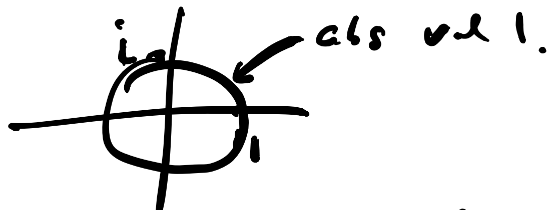
T preserves length.

So if $\|v\|=1$ then $\|T(v)\|=1$.

Preserves set of vectors of unit length

Every eigenvalue has abs. value 1.

Eigenvalues lie on <u>unit</u> circle in $\mathbb{C}$



Ex. 1×1 mx. $A=(a)$

Unitary $\iff \bar{a} = a^{-1} \iff a\bar{a} = 1 \iff |a|=1$

$a\bar{a} = |a|^2$, $|a|=1$ on unit circle

Unitary $\iff$ $|\text{eigenvalue}|=1$.

Ex. $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$$A^* = \bar{A}^t = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = A^{-1}$$

Real mx $A \in M_n(\mathbb{R})$;

unitary condition is $A^t = A^{-1}$!

Real unitary mxs: orthogonal mxs

(lin tr)

Name: preserve inn. prod.

$\Rightarrow$ " orthogonality

$v \perp w \Rightarrow Tv \perp Tw$.

rigid motion: rotations, reflections

Unitary mx: $A^* = A^{-1}$

$\underbrace{\qquad\qquad}_{\text{invertible}}$

Under $\cdot$: closed, assoc, id, inverse $\}$ group

unitary gp $\left\{ \begin{matrix} n\times n \text{ cx} \\ \text{unitary mxs} \end{matrix} \right\} = U(n) \subset GL_n(\mathbb{C})$

$\cup \qquad \cup$

orthog. gp $\left\{ \begin{matrix} n\times n \text{ real} \\ \text{orthog mxs} \end{matrix} \right\} = O(n) \subset GL_n(\mathbb{R})$

[Caution: Sometimes refer to a cx mx as "orthogonal" if $A^t = A^{-1}$. See H+K, p304. Different from unitary]

Thm (H+K, Thm 15, p312)

Let $T: V \to V$ be self adjoint operator on an inner prod spaceV. (so hermitian or symmetric if real)

a) Then every cx root of char. poly. $P_T(x)$ is a real #.

(ie all the eigenvalues are <u>real</u>)

b) Any two eigenvectors corresp to distinct eigenvalues are orthogonal. (not just lin ind)

Pf is an easy computation.

In 1×1 case:

$A = (a)$ is self adj.

$\iff A^* = A \iff \bar{a} = a \iff a \in \mathbb{R}$

the one eigenvalue is $a \in \mathbb{R}$.

<u>Thm</u> (Thm 18, p314, H+K)

(Spectral thm for self-adjoint operators)

Let $T: V \to V$ be a self-adjoint lin. op. on a fin dim inner prod sp V.

Then V has a <u>onb</u> consisting of eigenvectors of T.

Pf is by induction on dim, using prev result.

So: For a self-adjoint linear $T : V \to V$ ($\longleftrightarrow$ real sym mx, or cx Herm mx)

Can find a basis that's good in two ways:

1) o.n.b. ($\therefore$ basis is easy to work with)

2) puts T in diagonal form. (since basis of eigenvectors for T) ($\therefore$ T is easy to work with in this basis)

In particular:

Every real sym mx is diag'ble over $\mathbb{R}$. !

(More: wrt onb).

A real sym mx (or cx herm mx)

$\rightsquigarrow P_A(x) = \prod_{i=1}^{n} (x - a_i) \qquad a_i \in \mathbb{R}$

$$A \sim D = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \in GL_n(\mathbb{R})$$

wrt chg of basis from given onb (eg $e_1, \dots, e_n$) to a new onb.

Takes onb to onb; $\therefore$ preserves $\langle \, , \rangle$.

Chg of basis mx — — — —

so is unitary.

$$D = C^{-1} A C$$

C unitary : $C^{-1} = C^*$

$$\boxed{D = C^* A C}$$

D: diag, C: unitary

(easier to compute than C^{-1})

Say: A, D are unitarily equivalent.

Summary:

A $n \times n$ cx herm mx

$\Rightarrow$ A is unitarily equiv to a real diag. mx D.

A $n \times n$ real sym mx $\Rightarrow$ A is orthogonally equiv to ....D.

Does this carry over to other T, A?

Not in gen'l. — might not be diag'ble
— eigenvectors might not be $\perp$.

But — carries over (w/o $\mathbb{R}$ part of essential) to a class of T, A: normal.

$$AA^* = A^*A \quad (\text{Similarly for } T)$$

Thm (Spectral thm for normal operators)
(H+K, p317, Th 22 + Cor)

$A \in M_n(\mathbb{C})$ normal
$\Leftrightarrow$ A is unitarily equiv to a diag mx / $\mathbb{C}$.

(Pf: $\Rightarrow$: Every lin tr. / $\mathbb{C}$ is Δ'ble wrt onb; normal + Δ'r $\Rightarrow$ diag; $\Leftarrow$: UDU^{-1} normal)

unitary diag

Ex. A hermitian (incl real sym)
$\Rightarrow$ normal $\quad (A = A^*)$

Ex. A unitary (e.g. real orthogonal)
$(A^* = A^{-1})$

So: if A is unitary / $\mathbb{C}$
then $\exists$ onb of eigenvectors.
i.e. diag'ble wrt onb.

Ex. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in O(2) \subset U(2)$ ← eigenvalues: $i, -i$

$A \underset{\text{unitary}}{\sim} D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

↑ not real mx.

Caution: A is not real diag'ble.
(no real eigenvalues)

What if we work over other fields?
No notion of pos, pos def.
Chap 10, H & K.

$V \;/F$

Bilin form $\quad f : V \times V \to F$

$f(v,w) \in F$ gen'lizes inner prod

$\mathcal{B} = \{v_1, \_\_ v_n\}$ basis of V

$a_{ij} = f(v_i, v_j)$ $A = (a_{ij})$

mx of f wrt $\mathcal{B}$.

rank of A ("rank of f")

Say f is <u>non-degenerate</u> if

rank $=$ dim $= n$.

Equiv: $\forall v \neq 0 \ \exists w \neq 0 : f(v,w) \neq 0$

<u>True</u> for inn prod on real v s:

$v \neq 0 \Rightarrow v^{\perp} \neq V$. Take w not orthog to v.

<u>Symmetric</u> bilinear form:

$$f(v,w) = f(w,v)$$

$\Longleftrightarrow$ A is sym $(A = A^t)$

(true for real inn. prod.)

Say $v \perp w$ (wrt f)

if $f(v,w) = 0$.

f sym bil form

(gen'lizes real inn prod)

Defn $q(v) = f(v,v)$

(gen'lizes $\|v\|^2$) <u>quadratic form</u>

Ex. f on F^n be "dot product"

$v = (a_1 \ldots a_n)$

$w = (b_1 \ldots b_n)$

$f(v,w) = \sum a_i b_i$

$q(v) = a_1^2 + \cdots + a_n^2$

$$f \rightsquigarrow q$$

sym bil form	quad form	$f \leftrightsquigarrow q$
2 vbls	1 vbl	!

Conversely: can recover f from q:

Polarization identity: (gen'lizes P.I. / $\mathbb{R}$)

$$f(v,w) = \frac{1}{4} q(v+w) - \frac{1}{4} q(v-w)$$

unless $2=0$ in F ("characteristic 2")

V f.d. v.s. $/F$, $\operatorname{char} F \neq 2$

f sym bilin form on V.

Gram-Schmidt works as before

$\exists$ orthog basis $v_1 \dots v_n$ of V:

$f(v_i, v_j) = 0$ for $i \neq j$.

Equiv: $A = (a_{ij})$ $\quad a_{ij} = f(v_i, v_j)$

diagonal mx.

A+K, Th 3, p369

$$\begin{array}{c} r \text{ rows} \\ \neq 0 \end{array} \left[\begin{pmatrix} * & & & & & \\ & \ddots & & & & \\ & & * & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \right] \begin{array}{l} q(v_i) = 0. \\ n-r \end{array}$$

$r = \text{rank}$. $\qquad n \times n$

If f non-deg ($\operatorname{rk} = n$): no 0's.

Over $\mathbb{C}$: $q(v_i) = f(v_i, v_i)$

If $q(v_i) \neq 0$, replace v_i by $\dfrac{v_i}{\sqrt{q(v_i)}}$

a cx $\sqrt{\ }$

Get $\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{array}{l} \left.\right] r \\ \left.\right] n-r. \end{array}$

Classification f's $/ \mathbb{C}$

Over $\mathbb{R}$: $q(v_i) = f(v_i, v_i)$

If $q(v_i) > 0$, replace v_i by $\dfrac{v_i}{\sqrt{q(v_i)}}$

If $q(v_i) < 0$, replace v_i by $\dfrac{v_i}{\sqrt{|q(v_i)|}}$

$$\begin{array}{c} r\left\{\begin{matrix} \\ \\ \\ \\ \end{matrix}\right. \\ n-r\left\{\begin{matrix} \\ \\ \end{matrix}\right. \end{array} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & -1 & \\ & & & & & & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix} \begin{array}{c} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} r_1 \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} r_2 \\ \\ \\ \end{array}$$

$$\left.\begin{array}{l} r_1 + r_2 = r \\ r_1 - r_2 = s. \end{array}\right\} \text{Signature.}$$

(r, s, n) classifies f.

H+K, p379, Th 4+5.

Some applications

1) Rule for finding max/min for $f: \mathbb{R}^n \to \mathbb{R}$

— via Hessian mx

$$A = \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right)$$

At a crit pt: real sym. mx

$\therefore$ diag'ble

Translate crit pt to 0.

$$A \sim D = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$$

a_i's are eigenvalues

$$f = c + a_1 x_1^2 + \dots + a_n x_n^2 + \text{h.o.t.}$$

If all $a_i > 0$: min.
 eigenvals.

If all $a_i < 0$: max

If mixed: saddle pt.

If some $a_i = 0$: This test fails.

Special case $n=2$.
use rule for max/min of a fn of 2 vbls.

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2) Page Rank algorithm in Google
— using eigenvalues
Wikipedia

3) Singular value decomposition

— statistics,
signal processing,
weather forecasting, ...

"Diagonalize" an $m \times n$ mx

A $m \times n$ $\leftrightarrow T: \mathbb{R}^n \to \mathbb{R}^m$
$\shortparallel$
(a_{ij}) : $a_{ij} = 0$ unless $i = j$.

Make chg of vbls on $\mathbb{R}^n$ ← (can be diff
— — — — — $\mathbb{R}^m$. ← even if m=n)

Then can do this
— via algorithm that uses
pos def mx ($\leftrightarrow$ pos def f)
+ adjoints.

— Wikipedia