F field, scalars

V vs / F

$W \subset V$ is a subspace
subset

$\Longleftrightarrow$ $W \neq \emptyset$, & W is closed under $+$, sc. mult.

Ex. $\emptyset \neq S \subset V$ $\rightsquigarrow$
subset

$W = \{$all finite linear combs of elements of $S\}$

$$\sum_{i=1}^{n} a_i s_i, \quad a_i \in F, \; s_i \in S$$

This is a subspace

Subspace of V spanned by S

$W = \operatorname{span} S$.

This is the smallest subspace of V containing S.

(If $S \subset W' \subset V$, sub sp, then $W \subset W'$)

Ex. $W \subset V$ subspace

$\operatorname{span} W = W$

Ex. $S = \{(1,0,0), (0,1,0)\} \subset \mathbb{R}^3$

$\operatorname{span} S = x,y$ plane.

$\bigcap$ of any collection of subspaces of V is a subspace of V.

Ex. $W_1 \neq W_2 \subset \mathbb{R}^3$, planes

$W_1 \cap W_2 = $ line, subspace

Ex. Let $\phi \neq S \underset{\text{subset}}{\subset} V$

Take <u>all</u> the subspaces of V that contain S.

This is a subspace.

$= \operatorname{span} S$.

Other direction:

<u>Sum</u> of subspaces

$W_1, W_2 \subset V$ subspaces

$$W_1 + W_2 \overset{\text{def}}{=} \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$$

$\subset V$, subspace

$$W_1 \subset W_1 + W_2 \supset W_2$$

Ex. $V = \mathbb{R}^3$

$W_1 =$ x-axis, $W_2 =$ y-axis

$W_1 + W_2 =$ x,y-plane

$$W_1 + W_2 = \operatorname{span}(W_1 \cup W_2)$$

not usually a subspace

Linearly dependent vectors

$\underset{\text{set}}{S} \subset \underset{\text{v.s.}}{V}$ is <u>lin. dep.</u>

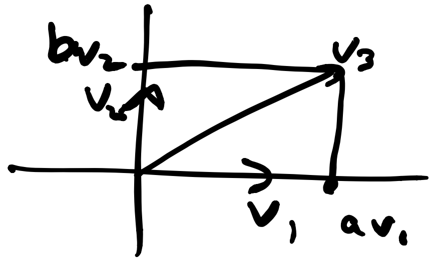
if some non-trivial finite linear combination of vectors in S equals 0.

Ex. $0 \neq v \in V$, $1 \neq c \in F$

$\{v, \underset{\text{"}v'\text{"}}{cv}\}$ $\quad cv - v' = 0$

lin. dep.

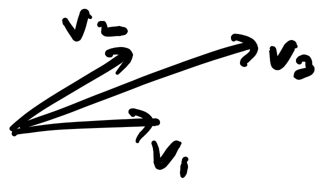
Ex. $S = \{v_1, v_2, v_3\} \subset \mathbb{R}^2$



$V_3 = av_1 + bv_2$

$av_1 + bv_2 - V_3 = 0$

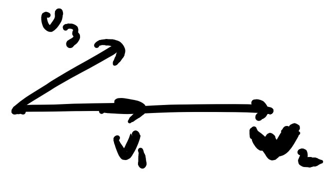
lin. dep.



use parallelogram

– Same conclusion

– lin dep.



lin dep.

3 vectors in $\mathbb{R}^2$: lin. dep.

---

3 vectors in $\mathbb{R}^3$:

$V_1 = (1, 0, 1)$, $V_2 = (0, 1, 1)$

$V_3 = (1, 1, 2)$

$V_3 = V_1 + V_2$ $\quad$ $V_1 + V_2 - V_3 = 0.$

lin. dep.

But $\underset{e_1}{(1, 0, 0)}, \underset{e_2}{(0, 1, 0)}, \underset{e_3}{(0, 0, 1)}$

not lin dep

— lin. independent

$ae_1 + be_2 + ce_3 = (a, b, c)$

$S \subset V$ is lin dep

$\Leftrightarrow$ one of the vectors in S is a lin comb. of finitely many of the others.

Ex. $\{e_1, e_2, e_3\} \subset \mathbb{R}^3$ lin ind

above: standard coord vectors

$\{e_1, e_2\} \subset \mathbb{R}^3$ lin ind.

$S' \subset S \subset V$

S lin ind $\Rightarrow S'$ lin ind

Ex. $0 \neq v \in V \Rightarrow \{v\}$ lin ind

Ex. $\{v, w\} \subset V$ lin ind
$v \neq w$

$\Leftrightarrow$ neither is a mult of the other

Ex. If $0 \in S \subset V$

then S is lin. dep.
$(1 \cdot 0 = 0)$

We will show:

a lin ind set in $\mathbb{R}^2$ has ≤ 2 elements.

- - — — $\mathbb{R}^3$ - - - 3 - -

- - - — — — $\mathbb{R}^n$ - - n - - -

In $\mathbb{R}^n$, have a set of n vectors that's lin ind

$e_1, \_, e_n$

$e_i = (0, \_, \overset{i\text{th}}{1}, \_, 0)$

We will prove:

If a set S of n vectors in $\mathbb{R}^n$ is lin ind, then S <u>spans</u> $\mathbb{R}^n$.

$\text{span}\, S = \mathbb{R}^n$

$(a_1, a_2, \_, a_n) = \sum_{i=1}^{n} a_i e_i$ unit coord vectors

$\{e_1, \_, e_n\}$ spans & is lin ind.

Say $S \subset V$ is a <u>basis</u> of V if it spans V & is lin. ind.

Ex $\{e_1, \_, e_n\} \subset V = \mathbb{R}^n$ is a basis — standard basis.

Ex. $W \subset \mathbb{R}^3$

$\}$ plane through 0 $\perp$ to $(1, 1, 1)$

$v \cdot (1,1,1) = 0$

$x + y + z = 0 \leftarrow v \perp (1, 1, 1)$

$v \cdot w = 0 \Leftrightarrow v \perp w$

$v = (a_1, \_, a_n)$, $w = (b_1, \_, b_n)$

$v \cdot w = \sum a_i b_i$.

$v = (1, 0, -1)$ $S = \{v, w\}$
$w = (0, 1, -1)$ lin. ind.

S spans:

$x \in W$ $x = (a, b, c)$ $a+b+c=0$
$= (a, b, -a, -b)$
$= av + bw$ ✓

S is a basis

W has a basis of two vectors.

Ex $P_5 = \{$ polys $f(x)$ of degree $\leq 5\}$

$a_0 + a_1 x + \dots + a_5 x^5$ $a_i \in F$

V.S.

Basis $\{1, x, x^2, x^3, x^4, x^5\}$
of 6 elements

Ex. $F = \mathbb{R}$

$W = \{$ sol'ns to $f'' - f = 0\}$

$= \{a e^{x} + b e^{-x} \mid a, b \in \mathbb{R}\}$

Basis: $\{e^{x}, e^{-x}\}$

Say a vs V is
finite dimensional if it has
a finite basis.

Above exs are fin. dim'l.

Ex. $\mathbb{R}^n$, P_5, plane $x+y+z=0$,
solns to the above DE,
3×4 matrices (basis of 12 elements)
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Ex. **Not** fin-dim'l:

$$P = \{\text{all polys}\}.$$

Has inf basis

$$\{1, x, x^2, x^3, \dots\}$$

Infinite dim'l.

We will see: all bases of a vs V have the <u>same</u> # of elements.

this #: dimension of V

$$\dim V$$

We will also show:

V has a finite basis

$\Leftrightarrow$ V has a finite spanning set

$$\left(\exists \text{ finite set } S \subset V \text{ s.t. } \operatorname{span} S = V \right)$$

Also: If $\dim V = n$ then:

1) Every lin ind set in V has $\leq n$ elements (e.g. ≤ 3 in $\mathbb{R}^3$)

2) Every spanning set for V has $\geq n$ elements (e.g. ≥ 3 in $\mathbb{R}^3$)

We will study relationships between different bases of V.

- change of basis

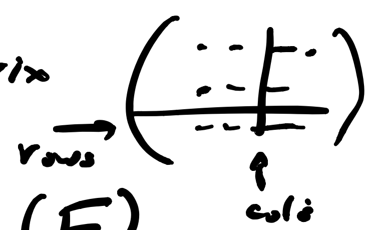
To study these:

use matrices

Quick review (H+K, Chap 1)

$m \times n$ matrix

m ← rows, n ← cols

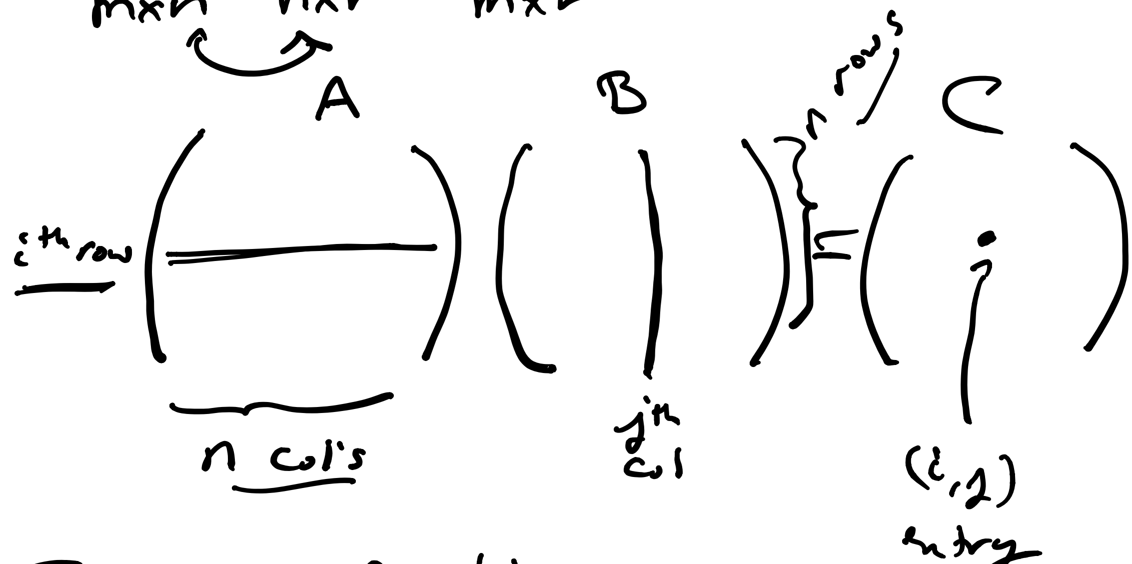


VS: $M_{m,n}(F)$ ← $\dim = mn$

basis: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Can add, scalar mult.

$$A \cdot B = C$$
$$m\times n \quad n\times r \quad m\times r$$



System of linear eqns
$\rightsquigarrow$ mult of mx's:

Ex. $x + 2y + z = 1$
$x + 2y + 2z = -3$
$2x + 4y + 2z = 2$

$$\rightsquigarrow \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

$$A \cdot X = B$$
$$3\times 3 \quad 3\times 1 \quad 3\times 1$$

In general:

homogeneous: RHS $= 0$

inhomogeneous: RHS $\neq 0$

To solve:

Can modify system of eq'ns

1. Subtract a mult of one row from another row.

2. Mult a row by non-0 scaler.

3. Interchange any two rows.

In terms of matrices

Row reduction

Gaussian elimination

For short –

Augmented matrices

$$\underset{A}{\begin{pmatrix} & & \end{pmatrix}}\underset{x}{\begin{pmatrix} \\ \end{pmatrix}} = \underset{B}{\begin{pmatrix} \\ \end{pmatrix}}; \quad \underset{\text{Aug. mx.}}{\left(A \,\middle|\, B\right)}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & 2 \end{pmatrix}}_{A}\Bigg|\underbrace{\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}}_{B} \xrightarrow[\text{from } R2,\,R3]{\text{subtract mult of } R1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

row echelon form

staircase

pivots

free variable

$$\xrightarrow[\text{of } R2 \text{ from } R1]{\text{subtract mult}} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

reduced row echelon form

Solve for pivot vbls in terms of free vbls.

$$\left.\begin{aligned} x + 2y &= 5 \\ z &= -4 \end{aligned}\right) \qquad \begin{aligned} x &= 5 - 2y \\ z &= -4 \end{aligned}$$

arbitrary

Line, not through 0

not a v.s.

(Inhomog.)

Instead, if we had 0's on RHS, we'd get a line through 0

— v.s.

In gen'l: Sol'ns form a v.s $\iff$ RHS $= 0$. (homog)

(Compare w PS1 #3) abt DE

2-12

m eqns in n unknowns

$m \times n$ matrix $\to$ $AX = B$

A: $m \times n$, X: $n \times 1$, B: $m \times 1$

Ex. If homog ($B = 0$), — so solns are v.s.
+ if $m < n$:

$\leq m$ pivots; $< n$ pivots

Some column has no pivot

So $\exists$ free vbl. $\therefore$ VS of solns $\neq \mathbf{0}$ v.s.

Have non-0 soln.

Ex. If $m = n$, A is $n \times n$ sq. mx.
(B not nec 0) Cases:

I: Red. row ech form of A has a row of all 0's:
$< n$ non-0 rows on left, $< n$ pivots, have free vbls.

a) If all 0's rows are $(0 \cdots 0 \mid 0)$ then $\exists$ free vbls > 1 soln.

b) If some row of 0's in A is $(0 \cdots 0 \mid c)$, $c \neq 0$: Inconsistent, no soln.

II Reduced row ech form of A has no row of 0's ? $n \times n$

Red row ech form $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{pmatrix} = I$

$n \times n$ id mx

$$C I = C = I C$$

$n \times n$

No free vbls; exactly 1 soln

i.e. 0 soln if homog.

For square matrices $(m=n)$

$$\underset{n\times n}{A}\,\underset{n\times 1}{X} = \underset{n\times 1}{B}$$

A might have an inverse A^{-1}

$$A A^{-1} = I = A^{-1} A$$

Then: $AX = B$, $\rightsquigarrow A^{-1}(AX) = A^{-1}B$

$$A^{-1}(AX) = (A^{-1}A)X = IX = X$$

$X = A^{-1}B$; solves system

A^{-1}? $n=2$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

A^{-1} exists $\iff ad - bc \neq 0$

$\underbrace{ad-bc}_{\Delta}$ determinants

$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$n=1$: $A = (a)$. $a \neq 0$. $A^{-1} = (a^{-1})$

bigger n?

2-14

More generally, to find inverses of $n \times n$ mx:

use row reduction:

Augment mx by n col's, using id. mx. I.

Reduce to get LHS $= I$:

Ex:

$$\underbrace{\begin{pmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}}_{A}\!\!\left|\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right) \xrightarrow[\text{of } R1 \text{ from } R2,\, R3]{\text{subtract multiple}} \left(\begin{matrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 3 & 2 \end{matrix}\right|\left.\begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}\right)$$

$$\xrightarrow[\text{of } R2 \text{ from } R3]{\text{subtract mult}} \left(\begin{matrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{matrix}\right|\left.\begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 3 & 1 \end{matrix}\right)$$

echelon form

$$\xrightarrow[\text{of } R2 \text{ from } R1]{\text{subtract mult}} \left(\begin{matrix} 2 & 0 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & -4 \end{matrix}\right|\left.\begin{matrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ -5 & 3 & 1 \end{matrix}\right)$$

$$\xrightarrow[\text{of } R3 \text{ from } R1,\, R2]{\text{subtract mult}} \left(\begin{matrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{matrix}\right|\left.\begin{matrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -5 & 3 & 1 \end{matrix}\right)$$

$$\xrightarrow[\text{by constants}]{\text{mult rows}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I}\!\!\left|\underbrace{\begin{matrix} \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{4} & -\frac{3}{4} & -\frac{1}{4} \end{matrix}}_{A^{-1}}\right)$$

A invertible $\Leftrightarrow$ Red row ech form is I.

Why does this work?

Each step in row red

$\Updownarrow$

Mult on left by elem. mx.

$R2 - 3 \cdot R1$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$2 \cdot R3$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$R1 \leftrightarrow R2$ $\quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$(A \mid I) \rightsquigarrow (E_1 A \mid E_1)$$

$$\rightsquigarrow (E_2 E_1 A \mid E_2 E_1) \rightsquigarrow \cdots$$

$$\rightsquigarrow (\underbrace{E_n \cdots E_1 A}_{=I} \mid E_n \cdots E_1)$$

$E_n \cdots E_1 = A^{-1}$ $\qquad A^{-1}$ ✓

A $\quad n \times n$ mx

row reduction

red. row. echelon form

R

Either

I $\quad R = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

Then A is invertible.

or

II $\quad R$ has a row of 0's.

Then R is not invertible.

$$\underbrace{\begin{pmatrix} * \\ 0 \cdots 0 \end{pmatrix}}_{R} \begin{pmatrix} * \end{pmatrix} = \begin{pmatrix} * \\ 0 \cdots 0 \end{pmatrix} \neq I$$

In fact, in II, A is not invertible

Reason: Row ops $\longleftrightarrow$ mult on left by elem mx

invertible

$$R = \underbrace{E_n \cdots E_2 E_1}_{\text{Invertible}} A$$

If A inv

then R would be inv.

This uses:
prod. of inv. mx's is inv.

$$(AB)^{-1} = B^{-1}A^{-1}$$

inv

$$A\underbrace{BB^{-1}}A^{-1} = AIA^{-1}$$
$$= AA^{-1} = I \checkmark$$

So: A is invertible

$\Longleftrightarrow$ red row ech. form of A is I.

$(*)$

$\Updownarrow$

A is row eq. to I

---

row ops $\longleftrightarrow$ left mult by elem. mx.

Say: A, B are <u>row equivalent</u>
if can pass from A to B
by row ops.

A, B row equiv $\Longleftrightarrow$

$$A = PB$$

prod. of elem mxs.

Let $B = I$

A is row eq to I $\overset{(**)}{\Longleftrightarrow}$

A = prod. of elem. mxs.

(*), (**):

<u>Thm</u> For a square mx A,

TFAE (the following are equivalent)

i) A is invertible

ii) A is row equiv to I

iii) A is a prod. of elem mx's.

(Th 12 §1.6 of H & K)

<u>Prop</u> A homog system $AX = 0$ (square)

has <u>only</u> the trivial sol'n.

$\Longleftrightarrow$ A is invertible.

Pf. $\Leftarrow$: $AX = 0$

$$A^{-1}AX = A^{-1}0$$

$$\underset{X}{\shortparallel} \qquad \underset{0}{\shortparallel}$$

$\Rightarrow$: $A \rightsquigarrow R$ red row ech form

No free vbls. $R = I$.

A is row eq to I.

Thm $\Rightarrow$ A invertible. ✓

A invertible:

$\exists B_{\approx A^{-1}}$ $AB = I = BA$

this is unique.

Reason: B, C both inverses of A.

$$C = CI = \underline{\underline{CAB}} = IB = B$$

Prop If A, B are $n \times n$ mxs, & $BA = I$ then $AB = I$.

So $B = A^{-1}$ and $A = B^{-1}$.

Pf. $BA = I$

$\Rightarrow$ the only soln to

$$AX = 0$$

is the trivial one.

(Left mult by B)
$\Downarrow$
$X = 0$

prev prop
$\Rightarrow$ A invertible; i.e. A^{-1} exists.

$$\Rightarrow AB = ABI = ABAA^{-1}$$
$$= AIA^{-1} = AA^{-1} = I$$

Or better:

$$BA = I$$
$$B = BAA^{-1} = A^{-1}$$

Prop Let A be $n \times n$ mx. Then:

A invertible $\Leftrightarrow$

$\forall$ col mx B ($n \times 1$), $\exists$! soln to $AX = B$.

Pf. $\Rightarrow$: Mult on left by A^{-1}
$\Leftarrow$: Solve with

$$B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

B_1 B_2 B_n

Get solns $X_1, X_2, \dots X_n$.

$$AX_i = B_i$$

$$X = (X_1 X_2 \cdots X_n)$$

$$AX = (B_1 \cdots B_n) = I$$

$\therefore$ by prev prop, A is invertible.

Back to v.s. V,
lin ind, span, basis:

Thm If a v.s. V has a basis consisting of m elements, then no lin. ind. set in V has $> m$ elts.

Ex. Can't have 3 lin ind vectors in $\mathbb{R}^2$
has a basis of 2 elts

Pf. Let the basis of m elts be $\beta_1, \ldots, \beta_m$.

Say S is a set of $> m$ elts.

Will show: S is lin dep.

In S, take $\alpha_1, \ldots, \alpha_{m+1}$.

STS these vectors are lin. dep.

β's are a basis $\Longrightarrow$

for each $j = 1, \ldots, m+1$,

$$\alpha_j = \sum_{i=1}^{m} \underbrace{a_{ij}}_{\text{in } F} \beta_i$$

To prove the thm, we will find a non-trivial lin. comb. of α's that's equal to 0.

i.e. non-trivial sol'n to eq'n

$$\longrightarrow \sum_{j=1}^{m+1} x_j \alpha_j = 0 \qquad x_j \in F$$

$$\sum_{j=1}^{m+1} x_j \alpha_j = \sum_{j=1}^{m+1} x_j \left(\sum_{i=1}^{m} a_{ij} \beta_i \right)$$

$$= \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{m+1} a_{ij} x_j \right)}\beta_i$$

want x_j's s.t. these are 0, for all i.

$$\begin{cases} \sum_{j=1}^{m+1} a_{1j} x_j = 0 \\ \vdots \\ \sum_{j=1}^{m+1} a_{mj} x_j = 0 \end{cases}$$

m eqns in $m+1$ unknowns. homog.

$\Rightarrow \exists$ free vbl

$\Rightarrow \exists$ non-trivial soln. $x_1, \dots x_{m+1}$

Done.

Cor If V is a f.d. v.s. ($\exists$ fin. basis.) then any two bases have the same # of elts.

Pf. Say bases $\mathcal{B}_1, \mathcal{B}_2$; m, n elts. (lin. ind.)

$\mathcal{B}_1$ basis of m elts $\Rightarrow n \le m.$

$\mathcal{B}_2$ - — n elts, $\Rightarrow m \le n$

$m = n.$ ✓

V f.d.v.s., all bases have same # of elts. Call this # the dimension

$\dim V$

Ex. $F = \mathbb{R}$ $\quad V = \mathbb{R}^n$

$\dim V = n.$

$\dim \mathcal{P}_5 = 6$

If V has a finite basis, then every basis is finite

Reason $\quad V \quad\quad B_1, \; B_2$

$\quad\quad\quad\quad\quad\quad n \quad\quad ?$

basis $\quad\quad$ lin ind.

So B_2 can't have $> n$ elts.

$\therefore B_2$ finite

Say V is an inf dim vs if some ($\therefore$ every) basis is infinite.

Ex. $\mathcal{P}$ $\quad 1, x, x^2, \text{—}$

Prop above says:
If $\dim V = n$ then
every lin ind set in V has $\leq n$ elts.
(no lin ind set has $> n$ elts)

Complement:
We will see:
If $\dim V = n$ then
every spanning set has $\geq n$ elts.
(no spanning set has $< n$ elts)

1st prove:
Prop If $S \subset V$ is a finite set,
then $\exists$ subset $T \subset S$
st T is lin. indep.
and $\operatorname{span} T = \operatorname{span} S$.

Ex. $V = \mathbb{R}^3$

$S = \{(1,0,0), (0,1,0), (1,1,0)\}$

lin comb of others

not lin indep.

Take $T = \{(1,0,0), (0,1,0)\}$

lin ind, same span.

delete

Pf of Prop

m elts

If $S \subset V$ is lin. ind,
then take $T = S$. ✓

If not (S lin deps),
then $\exists v \in S$ s.t.
v is a lin comb of the others
vectors in S.

Let $S_1 = S - \{v\}$. $m-1$ elts.

$\operatorname{span} S_1 = \operatorname{span} S$
(b/c v is a lin comb of the others).

If S_1 is lin ind, take $T = S_1$.

If not, repeat, get S_2 of $m-2$ elts.

This process terminates in $\leq m$ steps.
b/c S has m elts

So some S_i is lin ind,
take $T = S_i$. ✓

Cor 1 If $S \subset V$ is finite, & span $S = V$ then S contains a basis of V.

Pf. Apply Prop to S:

$\exists\, T \subset S$, T lin ind,

span T = span $S = V$

$\therefore T$ is a basis of V. ✓

Cor 2 If V has a finite spanning set, then V has a finite basis; i.e. V is a f.d.v.s.

Pf. By Cor 1, if we let S be a fin. sp. set, then S contains a basis, T. finite ($T \subset \overset{\text{finite}}{S}$) ✓

Nice characterization of f.d.v.s.

To gen'l'ze v.s.'s — allow more gen'l scalars

— ring — don't assume mult inverses

e.g. $\mathbb{Z}$ = {integers}

– "modules" (gen'lize vs's
— case of ring of scalars)

– Many results abt v.s's
fail for modules.

Ex. $\mathbb{Z}$, scalars.

Module: $\mathbb{F}_2 = \{0, 1\}$

$$\underset{\in \mathbb{Z},\ \text{scalar}}{3} \cdot \underset{\in \mathbb{F}_2}{1} = 1+1+1 = 1$$

$$\underset{\mathbb{Z},\ \neq 0}{2} \cdot \underset{\in \mathbb{F}_2}{1} = 1+1 = 0$$

Non-trivial lin comb $= 0$.

$\{1\}$ is lin. <u>dep</u>.

But it spans.

<u>No</u> basis. Cor 2 fails here.

<u>Cor 3</u> If $\dim V = n$, then
every spanning set for V
has $\geq n$ elts.

Equiv: If $S \subset V$ has $< n$ elts,
then S does not span V.

Pf. Say $S \subset V$, $\text{span}\, S = V$.

WTS S has $\geq n$ elts.

By Cor 2, S contains a basis T for V.

All bases of V have the same # of elts. So:

# of elts in T is n

$T \subset S$ $\geq n$ elts. ✓

We saw: every finite spanning set of a fdvs contains a basis. Complement:

<u>Thm</u> Every lin. ind set S in a fdvs V is <u>contained in</u> a basis.

Pf. Let $n = \dim V$.

$S = \{\alpha_1, \ldots, \alpha_m\}$ lin ind.

So $m \leq n$. (prev prop)

<u>If</u> S span V, then S spans & lin ind; $\therefore$ basis. ✓

If not (i.e. S doesn't span),
then span $S \neq V$.

Take $\alpha_{m+1} \in V$, $\alpha_{m+1} \notin S$. not in span S

Let $S_1 = S \cup \{\alpha_{m+1}\}$ lin ind
$= \{\alpha_1, \dots, \alpha_m, \alpha_{m+1}\}$
is lin ind (by PS2 #4)

If S_1 spans V, then S_1 is a basis $\supseteq S$, ✓.

Otherwise, repeat.

This terminates, b/c lin ind subset of n-dim VS has $\leq n$ elts.

Then S_i works. ✓