Recap: V, W vs's / F

lin. transf. $T: V \to W$

Image $(T) \subset W$ subspace
(range) dim = rank

kernel $(T) \subset V$ subspace
(null space) dim = nullity

rank T + nullity T = dim V

$T: V \to W$ lin transf.

Call T <u>one-to-one</u>
(injective) (<u>non-singular</u>)
if: distinct elts of V are sent
to " " " W.

Recall <u>Ex 1</u>: $T: \mathbb{R}^2 \to \mathbb{R}^2$
$T(v) = 3v$
this is injective.

<u>Ex 3</u> $T: \mathbb{R}^3 \to \mathbb{R}^2$
$T(a,b,c) = (a,b)$

$T(1,2,3) = T(1,2,4) = (1,2)$

Not inj.

(If $T: V \to W$ is injective, then $\ker T = 0$

($T(0) = 0$ for lin tr.)

Converse? Yes:

<u>Prop</u> $T: V \to W$ (lin transf) is inj $\iff \ker T = 0$

<u>Proof</u> ($\Rightarrow$): We saw.

($\Leftarrow$): Say $v \neq v'$ in V.

WTS $T(v) \neq T(v')$ in W.

$v \neq v'$, $\therefore v - v' \neq 0$

$\therefore T(v - v') \neq 0$ (by $\ker = 0$)

|| ← bc lin tr.

$T(v) - T(v')$

$T(v) \neq T(v')$. ✓

Ex. $T : \underline{\mathbb{R}^3} \longrightarrow \underline{\mathbb{R}^2}$

$(x, y, z) \longmapsto (a, b)$ where

$$\begin{array}{c}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \\ \begin{array}{ccc} T(e_1) & T(e_2) & T(e_3) \end{array}\end{array} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$A : \underline{2} \times \underline{3}$

$$AX = B$$

$T(x,y,z) = (x + 2y + 3z, \; 2x + 4y + 6z)$

$[T(v)] = A[v]$

$\operatorname{Ker} T = \{(x,y,z) \mid x + 2y + 3z = 0\} \subset \mathbb{R}^3$

plane, nullity $= 2 \longleftarrow 2$

not inj.

$\operatorname{Im} T = \{(a, 2a) \mid a \in \mathbb{R}\} \subset \mathbb{R}^2$ $\quad +$

line $\quad$ rank $= 1 \longleftarrow 1$

$\dim \mathbb{R}^3 = 3$

In general, if A is $m \times n$ matrix over F, get lin. fn. $T : F^n \to F^m$

$T(v) = w$ $\quad \underset{m\times n}{A} \underset{\underset{v}{n\times 1}}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \underset{\underset{w}{m\times 1}}{\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}}$

Write $T = T_A$.

$$T_A(e_j) = j^{\text{th}} \text{ col of } A$$

$$A\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

j^{th} slot

So: the col's of A are the images of the std basis vectors of F^n.

So: the col's of A span $\operatorname{image}(T_A)$.

$$\operatorname{rk}(T_A) = \dim(\operatorname{im} T_A)$$

$$= \dim \text{ of } \underbrace{\text{span of col's of } A}_{\text{column space of } A}$$

$$\stackrel{\text{def}}{=} \underline{\text{column rank}} \text{ of } A$$

$$\ker T_A = \text{sol'n space of eq'n } (AX=0)$$

$$\text{nullity } T_A = \dim \ker T_A$$

$$= \dim \text{ of soln sp of } (AX=0)$$

$$\text{nullity } T_A = n - \operatorname{rk} T_A, \quad n = \dim F^n$$

$$= \#\text{ of free vbls}$$

$$= \#\text{ of vbls} - \#\text{ of pivots}$$

$$= n - \#\text{ of non-0 rows in r.r.e.f.}$$

$$= n - \text{row rk } R$$

$$= n - \text{row rk } A$$

$$\therefore \operatorname{rk} T_A = \text{row rk } A.$$

$\operatorname{rk} T_A =$ col. rk of A (above)

$\qquad\qquad\qquad\ \ \|\ \longleftarrow\ !$

$\qquad\qquad\ =$ row rk of A

Call this rank of A.

A $m \times n$ matrix over F.

$\updownarrow$

$T_A : F^n \to F^m \quad [T_A(v)] = A[v]$

j^{th} col of A is $T_A(e_j)$

Turn this around:

Start with $T : F^n \to F^m$, lin tr.

Then: $T = T_A$ for some $m \times n$ matrix A.

What is A? It's the $m \times n$ mx whose j^{th} col is $T(e_j)$ $(j = 1, \dots, n)$

(bijection)

lin tr $T : F^n \to F^m$

$\updownarrow$ ← 1-1 correspondence

$m \times n$ matrices A over F

Algebra of lin transf's.

F field. V, W v.s.'s over F.

$S, T : V \longrightarrow W$ lin transf.

$$S+T : V \longrightarrow W$$
$$v \longmapsto S(v)+T(v)$$

$$(S+T)(v) = S(v)+T(v)$$

This is a lin tr.

$S : V \longrightarrow W$ lin tr, $c \in F$

$$cS : V \longrightarrow W$$
$$v \longmapsto cS(v)$$

$$(cS)(v) = c(S(v)).$$

This a lin tr.

Can add & scalar mult lin. tr's.

0-lin tr $0 : V \to W$

all $v \longmapsto 0$.

$$S+0 = S.$$

Set of all lin tr: $V \longrightarrow W$

$L(V, W)$, $Hom(V, W)$

This is a v.s. over F. (easy)

For $V = F^n$, $W = F^m$

every $T \in L(V, W)$ $\quad T: V \to W$

is of the form T_A $\quad A$ $m \times n$

Set of all $m \times n$ matrices over F.

$M_{m,n}(F)$; also a v.s.

Std basis

$E_{i,j}$: 1 in i,j entry

0 in other entries.

mn of these. $\dim M_{m,n}(F) = mn$.

$E_{ij} \rightsquigarrow T_{E_{ij}}$

$$T_{E_{ij}}(e_l) = \underbrace{l^{th} \text{ col of } E_{ij}}_{\substack{0 \text{ if } l \neq j \\ e_i \text{ if } l = j}}$$

$V \quad W$

$\downarrow \quad \downarrow$

$T_{E_{ij}}$'s form a basis of $L(F^n, F^m)$

$T \in L(F^n, F^m)$, $T = T_A$. $A = (a_{ij})$

$$T = \sum_{i,j} a_{ij} T_{E_{ij}}.$$

$\dim L(V,W)$

$\shortparallel$

mn.

Recap: V, W v.s. over F

$T \in L(V, W)$ $\quad T: V \longrightarrow W$ lin. tr.

v.s. over F

Ex. $V = F^n$, $W = F^m$

$A \in M_{m,n}(F) \rightsquigarrow T_A \in L(F^n, F^m)$

$\rightsquigarrow$ (*) $\quad [T_A(v)] = A[v]$

1-1 Corresp between $M_{m,n}(F)$, $L(F^n, F^m)$

Any $T \in L(F^n, F^m)$ is of the form T_A for some $A \in M_{m,n}(F)$:

A is the $m \times n$ mx over F whose j^{th} col. is $T(e_j)$, $j = 1, \dots, n$.

Corresp preserves $+$ and sc. mult:

$$A + B \rightsquigarrow T_{A+B} = T_A + T_B$$

$$cA \rightsquigarrow T_{cA} = cT_A$$

Bases of $M_{m,n}(F)$ & $L(F^n, F^m)$:

For $M_{m,n}(F)$:

$1 \le i \le m,\ 1 \le j \le n$

basis E_{ij} : 1 in (i,j) slot, 0 in others

Any $\underset{m \times n}{A} = (a_{ij}) = \sum_{i,j} a_{ij} E_{ij}$

$T_{E_{ij}} = E^{ij}$

$F^n \to F^m$ basis

$e_j \mapsto e_i$

$\underset{\text{other}}{e_\ell} \mapsto 0$

$T = T_A$

$= T_{\sum a_{ij} E_{ij}}$

$= \sum a_{ij} T_{E_{ij}}$

$= \sum a_{ij} E^{ij}$

$\therefore \dim L(F^n, F^m) = mn.$

V, W f.d.v.s.

dim n m

$\dim L(V,W) \overset{?}{=} mn$

V basis $\alpha_1, \dots, \alpha_n$

W " $\beta_1, \dots, \beta_m$

Take $E^{ij} \in L(V,W)$

E^{ij} takes α_j to β_i

" α_ℓ to 0 (others)

Again: get: these are a basis.

So: $\dim L(V,W) = mn$.

(Th 5, p75, H+K).

V, W, Z v.s.'s / F

lin tr $T : V \to W$

$U : W \to Z$

v $\qquad$ $T(v)$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[dr, "U \circ T"'] & W \arrow[d, "U"] \\
& Z
\end{tikzcd}$$

$U(T(v)) = U \circ T(v)$

$$\underbrace{U \circ T}(v) = U(T(v))$$

$U \circ T$ is also a lin tr.

(Easy to check)

Ex. $V = F^n$, $W = F^m$, $Z = F^p$

$T : V \to W \longleftrightarrow A \in M_{m,n}(F)$

$U : W \to Z \longleftrightarrow B \in M_{p,m}(F)$

$U \circ T : V \to Z \longleftrightarrow BA \in M_{p,n}(F)$

lin. tr.

$p,m \qquad m,n$

Recall: $T = T_A$, $U = T_B$

WTS $U \circ T = T_{BA}$ ✓

$$[T_{BA}(v)] = (BA)[v] = B(A[v])$$

$$= B[T_A(v)] = [T_B(T_A(v))]$$

$$= [T_B \circ T_A(v)]$$

$$= [U \circ T(v)]$$

$$T_{BA} = U \circ T$$

Composition of lin. tr. $\longleftrightarrow$ mx mult

__Ex__ $V = W$

$T \in L(V, V)$ $\qquad T: V \to V$

__linear operator__

__endomorphism__

$L(V, V) \overset{\text{def}}{=} L(V)$

$\quad \parallel \qquad\qquad \parallel$

$\mathrm{Hom}(V, V) = \mathrm{End}(V)$

If $V = F^n$, these corresp to $n \times n$ matrices

$A \in M_{n,n}(F) \overset{\text{def}}{=} M_n(F)$

In $L(V)$ can $+$, sc. mult
vs
__and__ compose.

In $M_n(F)$, can $+$, sc. mult,
vs
__and__ ~~mult.~~ matrices

Properties: v.s. properties and "mult":
associative, distrib over $+$,
$c(\alpha\beta) = (c\alpha)\beta = \alpha(c\beta)$
scalar; in $L(V)$ or $M_n(F)$

__algebra__ over F.

But no comm law of "mult"
(commutative)

Here: more: "mult" id.

Lin trs: $id_V : V \longrightarrow V$

$v \longmapsto v$

$$id_V \circ T = T = T \circ id_V$$

Mx's: $I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$$IA = A = AI.$$

$$T_I = id_V$$

An F-alg. with identity.

$A \in M_n(F)$

is an invertible mx: $\exists B \in M_n(F)$

st. $AB = I = BA$

Then B is unique. $B = A^{-1}$.

For lin trs: $T \in L(V)$ $T : V \to V$

Say T is an invertible lin tr: $\exists S \in L(V)$

st $S \circ T = id_V = T \circ S$.

Then: S is unique. $S = T^{-1}$.

$$M_n(F) \longleftrightarrow L(F^n)$$

$V = F^n$

$$A \longleftrightarrow T_A$$

$$\left(\begin{array}{l} T_{AB} = T_A \circ T_B \\ T_I = id_V \end{array} \right.$$

T_A is an invertible lin. tr.

$\Leftrightarrow$ A is an inv. mx.

then: $(T_A)^{-1} = T_{A^{-1}}$

More generally, $T: V \to W$ lin. tr.

$T \in L(V, W)$

Say T is an <u>inv. lin. tr.</u> if

$\exists\, S: W \to V$ $\qquad S \in L(W, V)$

st

$(*)$ $S \circ T = id_V$ and $T \circ S = id_W$

$$V \xrightarrow{T} W \xrightarrow{S} V \qquad W \xrightarrow{S} V \xrightarrow{T} W$$

Say $T: V \to W$ is any lin. tr.

Say $S: W \to V$ is any map
(not assumed linear)

Suppose (*) holds.

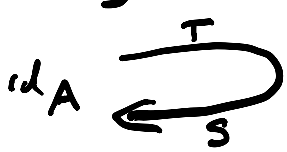
Then: S must be linear.

[Easy to check.
See H+K, p 79, Th 7]

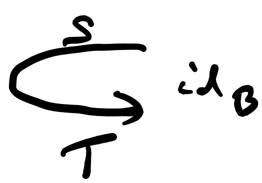
When is a lin. tr. invertible?

More generally, say A, B are sets.

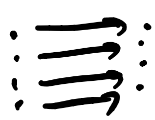
$$A \underset{\exists ? S}{\overset{T}{\rightleftarrows}} B \quad : \quad S \circ T = \mathrm{id}_A \quad \leftarrow$$



$$T \circ S = \mathrm{id}_B \quad \leftarrow$$

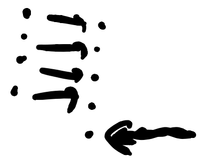


T undoes S
& S " T.



(one-one)

 not inv. b/c not injective



" " " (onto)
not surjective

1-1 & onto (1-1 correspondence

[inj + surj (bijection)]

$A \xrightarrow{T} B$

$S(b) = a$ if a is the unique elt of A s.t. $T(a) = b$.

$T: A \to B$ is an invertible map of sets

$\iff T$ is bijective.

Back to v.s. & lin. tr.

$T: V \to W$

To be invertible lin tr, must be bijective.

– & conversely, since an inverse of a lin transf is automatically . - — — .

i.e. lin tr. is invertible $\iff$ bijective

$T: V \to W$

surj? Take arb. elt $w \in W$
See if $w = T(v)$ some $v \in V$.

inj? Iff $\ker T = 0$

$T: V \to W$ is inj

$\Updownarrow$

T takes lin ind subsets of V
to - - - - - - W.

(Th 8 p 80)

$T: V \to W$ surj

$\Updownarrow$

T takes spanning sets in V
to - - —— W.

$T: V \to W$ bij $\Longleftrightarrow$ T is an inv. lin. tr.

$\Updownarrow$ $\Uparrow$

T takes basis of V
to - - - - - W.

T is an isomorphism

---

rk + nullity = dim

$\Downarrow$

Th (Th 9, p81, H+K)

Say V, W are f.d. v.s.

+ $\dim V = \dim W$,

+ $T: V \longrightarrow W$.

TFAE:

i) T is invertible (bij, iso)

ii) T is non-singular (inj, one-to-one)

iii) T is onto (surjection)

False if $\dim V \neq \dim W$

Ex. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(x, y) \longmapsto (x, y, 0)$$

inj, not surj, not bij

Ex. $S: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$(x, y, z) \longmapsto (x, y)$$

surj, not inj, not bij.

Say $T: V \to W$ is iso.

V, W are f.d.v.s.

$\Rightarrow \dim V = \dim W$

So: two fdvs's $/F$ are iso

$\Leftrightarrow$ same dim

$\Leftarrow$: V $\{\alpha_1, \dots, \alpha_n\}$ bases

W $\{\beta_1, \dots, \beta_n\}$

$\alpha_i \mapsto \beta_i$

iso.

$\sum a_i \alpha_i \mapsto \sum a_i \beta_i$

Ex. P_5 Basis $1, x, x^2, x^3, x^4, x^5$

$\mathbb{R}^6$ " $e_1, e_2, \dots, e_6$

dim: 6. iso.

$$V \xrightarrow{\sim} W$$

If $\exists$

$T: V \to W$ iso, say V, W are <u>isomorphic.</u>

$\dim V = n$. $\dim F^n = n$

$\therefore$ V is isomorphic to F^n.

Ex. $V = F^n$, $W = F^m$

$L(V, W)$ $\qquad$ $M_{m,n}(F)$

mn $\qquad$ mn

bij lin.tr., ∴ <u>iso</u>

$M_{m,n}(F) \longrightarrow L(V, W)$

$A \longmapsto T_A$

$T_{A+B} = T_A + T_B$

$T_{cA} = cT_A$

$M_{m,n}(F) \xrightarrow{\sim} L(V, W)$ $\quad$ <u>iso</u> of vs's

<u>mn</u> $\rightsquigarrow mn$

Say $m = n$, $V = W = F^n$.

$L(V, V) = L(V)$ $\qquad$ vs's of dim n^2

$M_{n,n}(F) = M_n(F)$ $\qquad$ $+$, sc. mult.

Add'l op:

On $M_n(F)$: mx mult AB

On $L(V)$: composition $\quad$ & on $T_{AB} = T_A \circ T_B$

$M_n(F) \longrightarrow L(F^n)$
iso of algebras over F.
w identity

$T_I = id_{F^n}$.

$\{$inv. mx's$\} = GL_n(F) \subset M_n(F)$
not a v.s. algebra
No 0.

$A, B \in GL_n(F) \Rightarrow AB \in GL_n(F)$

$(AB)^{-1} = B^{-1}A^{-1}$

$GL_n(F)$: mult law
assoc; identity; inverse
I A A^{-1}

group

general linear group
in dim n
(not nec. comm)
(GL_n is comm $\Leftrightarrow n=1$)

Ex. V vs $/F$

$L(V)$ v.s. endo-morphisms
$\cup$ auto-morphisms
$\mathrm{Aut}\, V = \{$invertible lin tr$\}$
not a v.s. $0 \notin \mathrm{Aut}\, V$.
lin tr

Have composition law on
Aut V:

$S, T \in \mathrm{Aut}\, V$

$S \circ T \in \mathrm{Aut}\, V$

b/c $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$

This a group.

If $V = F^n$:

$$A \longmapsto T_A$$

$$\begin{array}{ccc} M_n(F) & \xrightarrow{\sim} & L(V) \\ \cup & & \cup \\ GL_n(F) & \xrightarrow[\text{gp. iso.}]{\sim} & \mathrm{Aut}(V) \end{array}$$

$\rightsquigarrow$

$$A \longmapsto T_A$$

$$A \text{ inv} \Longrightarrow T_A \text{ inv}$$

$$A^{-1} \longmapsto T_{A^{-1}} = (T_A)^{-1}$$

b/c $T_{AB} = T_A T_B$