Quotient space

Appendix A.4, p.394

$T: V \to W$ lin. fn.

Let $N = \ker T$ (nullspace)

Let $w_0 \in W$.

If $w_0 \in \operatorname{im} T$, then

$w_0 = T(v_0)$, $v_0 \in V$.

1) If $n \in N$, then

$$T(v_0 + n) = T(v_0) + T(n)$$
$$= w_0 + 0 = w_0.$$

2) Conversely, if $v \in V$ and $T(v) = w_0$

then $v = v_0 + n$ for some $n \in N$.

Reason: $T(v - v_0) = T(v) - T(v_0)$
$$= w_0 - w_0 = 0$$

$\therefore \underbrace{v - v_0}_{n} \in N$; $v = v_0 + n$. ✓

So: $\{v \in V \mid T(v) = w_0\} = \{v_0 + n \mid n \in N\}$

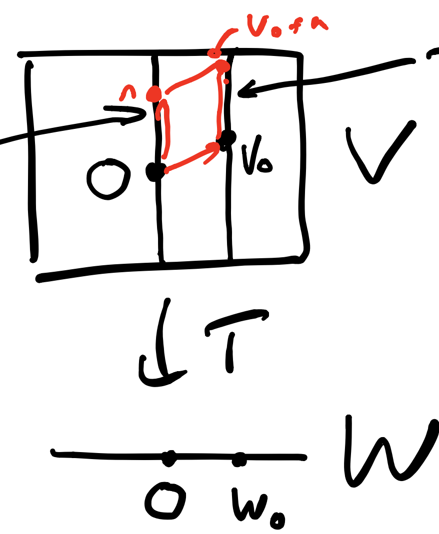
$$\underbrace{T^{-1}(w_0)}_{\substack{\text{inverse image} \\ \text{(preimage) of } w_0}} \overset{\text{def}}{=} \{v \in V \mid T(v) = w_0\} = \{v_0 + n \mid n \in N\} \overset{\text{def}}{=} \underbrace{v_0 + N}_{\substack{\text{coset} \\ \text{of } N \text{ in } V}}$$

In particular, take $v_0 = 0 \in V$
$w_0 = T(v_0) = 0 \in W$

$$\underset{w_0}{T^{-1}(0)} = \underset{v_0 \in V}{0} + \underset{\ker T}{N} = N$$

Every elt $v_0 \in V$ is in one of these cosets! $v_0 \in v_0 + N$

Picture



Ex. $V = \mathbb{R}^2$, $W = \mathbb{R}$, $T(x, y) = x$ projection onto x-axis

$N =$ y-axis. $T^{-1}(x_0) = \{(x_0, y) \mid y \in \mathbb{R}\}$
$w_0 = x_0 \in W$ $\qquad = (x_0, 0) + N$, $v_0 = (x_0, 0)$

Ex. $V = \mathbb{R}^3$, $W = \mathbb{R}^2$

$T(x, y, z) = (x, y)$. proj onto $x_T y$ plane.

$N = z$-axis

$$T^{-1}(\underbrace{x_0, y_0}_{w_0}) = \{(x_0, y_0, z) \mid z \in \mathbb{R}\}$$

$$= (\underbrace{x_0, y_0, 0}_{v_0}) + N$$

$w_0 \xleftarrow{T} v_0$

The cosets of N in V:

these are not subspaces of V,

exc. for $N = 0 + N$

b/c they don't contain 0.

If $v_0 + N$, $v_1 + N$ are two cosets,

then either $=$ or disjoint.

cosets of N partition V.

Reason: If $v_1 \in v_0 + N$ then $v_1 + N = v_0 + N$.

If $v_1 \notin v_0 + N$, $v_1 - v_0 \notin N$

$\therefore$ no elt in $v_1 + N$ is in $v_0 + N$; so disjoint

Ex. System of m lin eqns in n unknowns over F.

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

$A = (a_{ij})_{m \times n} \longleftrightarrow$ lin tr.

$$T : F^n \to F^m$$
$$T(x_1, \dots, x_n) = (b_1, \dots, b_m)$$
$$\Updownarrow$$
$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{B}$$

where $B = 0$

{Solns to homog system}

$$\ker T \overset{\shortparallel}{=} N = T^{-1}(0)$$

Pick $B \neq 0$. Inhomog system.

{solns to this inhomog system} $\ni \bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ (a particular soln)

$$\overset{\shortparallel}{T^{-1}}(b_1, \dots, b_m)$$
$$\shortparallel$$
$$\underbrace{(\bar{x}_1, \dots, \bar{x}_n)}_{\text{partic. soln to inhomog sys.}} + \underbrace{N}_{\text{gen'l soln to homog system.}}$$

Ex. Solns to lin ODE's.

$$y'' + y = x^3 + 6x$$

V = vs of infinitely diff'ble fns on real line.

$$T : V \longrightarrow V$$
$$y \longmapsto y'' + y$$

$\ker T = N$ = gen'l soln to homog diff eqn $y'' + y = 0$

2 dim'l vs.

= span of $\cos x$, $\sin x$

A particular soln to given ODE:

$y = x^3$. $T(x^3) = x^3 + 6x$

$\therefore$ gen'l soln to inhomog DE

$y'' + y = x^3 + 6x$ is

$$T^{-1}(x^3 + 6x) = x^3 + \underbrace{\ker T}_{N}$$

$$= \{x^3 + a\cos x + b\sin x \mid a, b \in \mathbb{R}\}$$

$$= y_p + y_h$$

y_h gen'l soln to homog eqn.

y_p partic soln to inhomog eqn

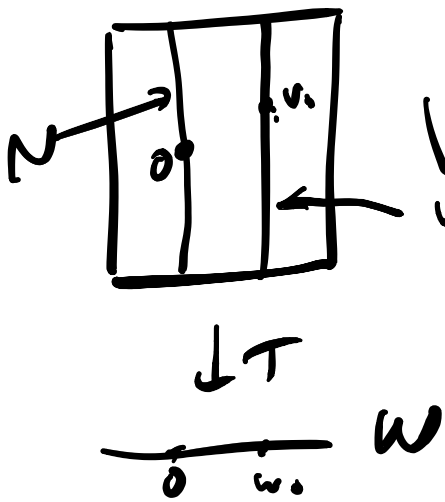
Say $V \xrightarrow{T} W$ surjection.

Let $N = \ker T$.

So $\forall w_0 \in W \;\; \exists v_0 \in V$

st $T(v_0) = w_0$.

$$T^{-1}(w_0) = v_0 + N$$



$$\begin{array}{ccc} \{\text{cosets of } N \text{ in } V\} & \ni & v_0+N \\ \text{bij} \downarrow & & \downarrow \\ W & \ni & w_0 = T(v_0) \end{array}$$

Can add cosets: well defined

$$\begin{array}{ccccc} (v_0+N) & + & (v_1+N) & \overset{def}{=} & (v_0+v_1)+N \\ \downarrow & & \downarrow & & \downarrow \\ w_0 & + & w_1 & = & w_0+w_1 \end{array}$$

Can scalar mult:

$$\begin{array}{ccc} c(v_0+N) & \overset{def}{=} & cv_0+N \\ \downarrow & & \downarrow \\ cw_0 & & cw_0 \end{array}$$

$\{$ cosets of N in $V \}$ is a vector space

$\Big\downarrow$ iso of VS's.

$\ker T \qquad W \qquad$ - - - -

$$\overset{\ker T}{\overset{\parallel}{N}} \subset V \xrightarrow{T} W \quad \text{surj lin tr}$$

$$\{\text{cosets of } N \text{ in } V\} \xrightarrow{\sim} W$$

$$\parallel$$

$$V/N$$

"V mod N"

quotient of V by subspace N.

$$V/N \xrightarrow{\sim} W$$

1st iso thm.

$T : V \longrightarrow W$ lin tr.

$$V \twoheadrightarrow \operatorname{im}(T) \qquad V/N$$

surj $\qquad N = \ker T$

$\operatorname{Im}(T)$ is a quotient of V.

Conversely, every quotient of V is the image of a lin. tr. on V:

$N \subset V$ any subsp. $\qquad V/N = \{v + N \mid v \in V\}$ v.s.

$$V \longrightarrow V/N \qquad \text{lin. tr.}$$
$$v \longmapsto v + N$$

ker $= \{v$ that map to $0 + N = N\}$
$= \{v \in N\} = N.$

Ex. $V = \mathbb{R}^3 \qquad W = \mathbb{R}^2$

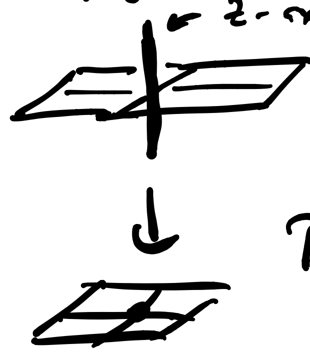
$T : V \longrightarrow W$

$(x, y, z) \longmapsto (x, y)$ proj onto x,y plane

$\ker T = N =$ z-axis

$\operatorname{im} T =$ x,y plane $= W$ (surj)

$$\begin{tikzcd} P \arrow[r, phantom, "\subset"] \arrow[dr, "\text{iso}"'] & V \arrow[d, "T"] \\ & W \end{tikzcd}$$



$P \xrightarrow{\sim} W \approx V/N$

$N+P = V = \mathbb{R}^3$ $\qquad (x,y,z) = (0,0,z) + (x,y,0)$

z-axis $= N \qquad P = x,y$ plane

$(0,0,z) \qquad (x,y,0) \qquad (x,y)$

$0 \xleftarrow{} O \xleftarrow{T} N \qquad N \cap P = O \qquad P \xrightarrow[\sim]{T} \mathbb{R}^2 = W$

$\dim N = 1, \quad \dim P = 2, \quad \dim V = 3$

$1+2=3.$

$\dim N + \dim P = \dim V$

$$\underset{\substack{\| \\ N+P}}{\dim V} = \dim N + \dim P - \underbrace{\dim (N \cap P)}_{0 \,\to\, O}$$

$$P \xrightarrow{\sim} W \cong V/N, \qquad P \subset V = \mathbb{R}^3 \xrightarrow{T} W, \qquad W = \mathbb{R}^2$$

"Can view P as the same as W and V/N via these iso's"

More generally:

V v.s. $\qquad$ subsp's P, Q

If $V = P+Q$, + if each $v \in V$ is uniquely $p+q$ for $p \in P$, $q \in Q$, we $P \oplus Q$

Say: P, Q are complementary subspaces of V.

Equiv: $V = P + Q$ and $P \cap Q = 0$. compl.

Write: $V = P \oplus Q$

PS 7 #4b.

$T: V \twoheadrightarrow W$, $N = \ker T$,

$P \subset V$ subsp. Then:

Restrict. of T to P $\quad (P \xrightarrow{T} W)$ is iso

$\Updownarrow$

$P + N$ are complements $(V = P \oplus N)$

(H + K, App A.4, Thm on p 396)

$W = V/N$

Say $T: V \twoheadrightarrow W$, $N = \ker T$, $W = V/N$

$\Longrightarrow \exists P \subset V$ st $V = N \oplus P$ (complement)

and T restr. to P is iso.

Converse

f.d.v.s.

Reason: Take basis of N:

$v_1, \ldots v_n \qquad n = \dim N.$

Extend to basis of V: $v_1, \ldots v_n, \ldots v_d$

$d = \dim V$

Let $P = \text{span}\{v_{n+1}, \dots, v_d\}$.
Check: $V = N \oplus P$,
$P \xrightarrow{I} W$ iso ✓

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§3.7 Transpose of a lin. tr.

$T: V \longrightarrow W$ lin tr. of f.d.v.s.
dim n dim m

$\mathcal{B}$ $\mathcal{C}$ Bases

$T \longleftrightarrow A$ $m \times n$ matrix

$T(v) = w \longleftrightarrow A[v]_{\mathcal{B}} = [w]_{\mathcal{C}}$

Ex. std bases on $\mathbb{R}^n$, $\mathbb{R}^m$

A $m \times n$ $A = (a_{ij})$

transpose A^t $n \times m$ $A^t = (a_{ji})$

$$A^t_{ij} = A_{ji}$$

row space of A = col space of A^t
col - - - = row - ——

What lin trans. corresp to A^t?
(from m-dim'l vs to n-dim'l vs)

Ans. $\quad W^* \longrightarrow V^*$ dual spaces

dual bases $\quad \mathcal{C}^* \quad \mathcal{B}^*$

$$\begin{tikzcd}
V \arrow[rr, "T"] \arrow[dr, "g\circ T"'] & & W \arrow[dl, "g \in W^*"] \\
& F &
\end{tikzcd}$$

V^*

Get lin tr: $W^* \longrightarrow V^*$

$g \longmapsto g\circ T$

T^*, T^t

The mx assoc to $T^*: \underset{\mathcal{C}^*}{W^*} \longrightarrow \underset{\mathcal{B}^*}{V^*}$

is A^t. $\quad T^*(g) = g\circ T$

H+K, §3.7, Th 23

$T: \underset{\substack{\mathcal{B}\\ n}}{V} \longrightarrow \underset{\substack{\mathcal{C}\\ m}}{W} \quad \longleftrightarrow \quad \underset{m\times n}{A}$ $\quad$ row rk A = col rk A = rk A

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ||\qquad\qquad\qquad ||$

$T^*: \underset{\substack{\mathcal{C}^*\\ m}}{W^*} \longrightarrow \underset{\substack{\mathcal{B}^*\\ m}}{V^*} \qquad \underset{n\times m}{A^t}$ $\quad$ col rk A^t = row rk A^t = rk A^t

$W \supset \operatorname{im} T = $ col sp of A

$\dim(\operatorname{im} T) = \operatorname{rk} T = \operatorname{rk} A$

$V \supset \ker T = $ soln sp to homog eqn $(AX=0)$

$\dim(\ker T) = \operatorname{nullity}(T) = n - \operatorname{rk} T$

$V^* \supset \operatorname{im} T^* = $ col sp of A^t $=$ row sp of A

$\dim(\operatorname{im} T^*) = \operatorname{rk} T^* = \operatorname{rk} A^t$

$= \operatorname{rk} A = \operatorname{rk} T$

$W^* \supset \ker T^* = $ soln sp to homog eqn $(A^t Y = 0)$

$\dim(\ker T^*) = \operatorname{nullity}(T^*) = m - \operatorname{rk} T$

4 fundamental subspaces
assoc. to T (or to A).