Chap. 6

Eigenvalues, eigenvectors, diagonalization, triangularization

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$T: V \to V \rightsquigarrow A \leftrightarrow T$

$\mathcal{B} \quad \mathcal{B}$ basis

mx

$T = T_{A,\mathcal{B}}$

Change basis $\rightsquigarrow$ change mx of T

mx A, B wrt two bases

$\Leftrightarrow$ A, B are <u>similar</u> $A \sim B$

$B = C^{-1}AC$ for some C

Some mx's are easier to work with than others.

e.g. to compute det

— — — rank

. — — —

Given T:

Does there exist a basis of V for which the mx of T is diagonal? — or at least, triangular?

Diagonal:

$T: V \to V$ n dim'l

$\mathcal{B}$ basis, $\leadsto$ mx $A \in M_n(F)$

Is there a diagonal mx $D \sim A$?

If so, what is D? What is the new basis?

If $A \sim D = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$

rep's T wrt $v_1, \ldots v_n$

Then $T(v_j) = c_j v_j$.

If we find lin ind vectors $v_1 \ldots v_n$ st $T(v_j) = c_j v_j$ for some $c_j \in F$

then $D = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$ rep's T wrt $\{v_1, \ldots v_n\}$

$A \sim D$.

Then $\det A = \det D = \prod c_i$

$\operatorname{rk} A = \operatorname{rk} D = \#$ of non-0 c_i's.

$T: V \to V$

If $v \in V$, $c \in F$, $T(v) = cv$

call v an eigenvector of T.

("characteristic vector")

If $v \in V$ is a non-0 eigenvector

with $T(v) = cv$ $c \in F$:

we call c the corresponding

eigenvalue of T.

("characteristic value")

So to diag mx $A \overset{n\times n}{\leftrightarrow} \underset{\mathcal{B}}{T}$ n dim'l:

Find n lin ind eigenvectors v_i for T

& their corresp eigenvalues c_i.

Then: $A \sim D = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$

wrt basis $v_1 \ldots v_n$ of V.

How to find the eigenvectors $v = (x_1, \ldots, x_n)$

& eigenvalues c?

A $n \times n$. $A[v] = c[v]$

Solve $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = c\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Equiv: $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = cI\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

" $\underbrace{(A - cI)}_{n \times n}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$ $\quad \nwarrow \neq 0$

$\exists$ non-0 soln $\iff \det(A - cI) = 0$

This depends on c.

If this happens, c is an eigenvalue

& $(x_1, \ldots x_n) = v$ is the corresp. eigenvector.

Ex. $A = \begin{pmatrix} 1 & 7 \\ 4 & 4 \end{pmatrix} \leftrightarrow T: \mathbb{R}^2 \to \mathbb{R}^2$

$A - cI = \begin{pmatrix} 1-c & 7 \\ 4 & 4-c \end{pmatrix}$

$$\begin{aligned} \det(A - cI) &= (1-c)(4-c) - 7 \cdot 4 \\ &= c^2 - 5c - 24 \\ &= (c-8)(c+3) \end{aligned}$$

$\det = 0 \iff c = 8, -3. \leftarrow$ eigenvalues.

$A \sim D = \begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}$ if $\exists$ basis of eigenvectors.

(Here, $v_1 \leftrightarrow 8$, $v_2 \leftrightarrow -3$,

$T(v_1) = 8v_1$, $T(v_2) = -3v_2$

$\therefore$ v_1, v_2 are not mult's of each other; lin ind. in $\mathbb{R}^2$

$\therefore$ form a basis. $\therefore$ $A \sim D$.

Find v_1, v_2:

$\exists$ non-0 sol'ns to $(A - cI)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Find!

pivot free

c = 8: $A - cI = \begin{pmatrix} -7 & 7 \\ 4 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$x - y = 0$ $\qquad x = y$ $\qquad (1, 1)$ & multiples

Take $v_1 = (1, 1)$. $\leftrightarrow 8$

free

c = -3: $A - cI = \begin{pmatrix} 4 & 7 \\ 4 & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 7/4 \\ 4 & 7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 7/4 \\ 0 & 0 \end{pmatrix}$

Sol'n's to homog eq'n $\quad x + \frac{7}{4}y = 0$

$y = -\frac{4}{7}x$

$v_2 = (7, -4)$ is a sol'n

(also its multiples)

-3

v_1, v_2 are lin. ind.; basis of $\mathbb{R}^2$

In this basis: mx of T is $D = \begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}$

$A \sim$

Change of basis mx C:

Express v_1, v_2 in terms of e_1, e_2

Columns of C

$v_1 = (1,1) = e_1 + e_2$

$v_2 = (7,-4) = 7e_1 - 4e_2$

$$C = \begin{pmatrix} 1 & 7 \\ 1 & -4 \end{pmatrix} \qquad D = C^{-1}AC$$

$$\begin{array}{cc} \uparrow & \uparrow \\ v_1 & v_2 \end{array}$$

(check)

In general: $A = (a_{ij}) \in M_n(F)$

$$\updownarrow$$

$$T : F^n \to F^n$$

To find eigenvalues c

& eigenvectors v of A.

— if $\exists$ basis of F^n

consisting of eigenvectors of T

then diagonalize A.

Want c st. $\det(A - cI) = 0$

$\iff \det(cI - A) = 0$

$\det(xI - A) \quad \in F[x]$, deg $= n$ monic

$$= \det\begin{pmatrix} x-a_{11} & -a_{12} & \cdots & -a_{1n} \\ \vdots & x-a_{22} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & \cdots & x-a_{nn} \end{pmatrix}$$

<u>Characteristic poly of A</u>, $P_A(x)$

(*) Eigenvalues of A are the roots of $P_A(x)$. $\in F[x]$

monic, degree n

$P_A(x) = 0$ for $x = c$,

eigenvects

(**) $(A - cI)X = 0$ has non-0 solns

eigenvalue

$X = [v]$

$A[v] = c[v]$. Find c: roots of $P_A(x)$.

Find v: solns to (**).

A poly of degree n has $\leq n$ roots, in F.

Say: $P_A(x)$ <u>has</u> n roots in F

distinct.

$c_1, \ldots, c_n$; get $v_1 \ldots v_n$

eigenvalues   Corresp eigenvectors

We would like to use $v_1, \ldots v_n$ as a basis of eigenvalues of F^n.

Then: wrt this basis, mx of T is $D = \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix}$

How do we know if $v_1 \dots v_n$ are a basis?

Iff $v_1 \dots v_n$ are lin. ind.

<u>Prop</u> If $v_1, \dots v_k$ are non-0 eigenvectors for $A \in M_n(F)$ corresp to distinct eigenvalues $c_1, \dots c_k$, then $v_1 \dots v_k$ are lin ind.

$\therefore$ If we have $v_1, \dots v_n$ as above, then a basis; A is <u>diagonalizable</u>.

<u>Pf of Prop</u>

If not always true, take a minimal counterexample (smallest k)

$c_1, \dots, c_k \in F$ distinct eigenvalues

$v_1, \dots, v_k$ eigenvectors for A $n \times n$ mx.

So: $v_1, \dots v_k$ are lin. dep.

(*) $a_1 v_1 + \cdots + a_k v_k = 0$ $\quad a_i \in F$, not all 0.

Since this is minimal, **no** $a_i = 0$. (or: could omit it)

$\underset{n\times n}{A} \leftrightarrow T = T_A : F^n \to F^n$

Apply $T, T^2, \ldots, T^{k-1}$ to (*)

$$\left[\begin{array}{l} a_1 v_1 + \cdots + a_k v_k = 0 \\ a_1 c_1 v_1 + \cdots + a_k c_k v_k = 0 \\ a_1 c_1^2 v_1 + \cdots + a_k c_k^2 v_k = 0 \\ \vdots \\ a_1 c_1^{k-1} v_1 + \cdots + a_k c_k^{k-1} v_k = 0 \end{array}\right.$$

$$\begin{pmatrix} a_1 & \cdots & a_k \\ a_1 c_1 & \cdots & a_k c_k \\ a_1 c_1^2 & \cdots & a_k c_k^2 \\ \vdots & \ddots & \\ a_1 c_1^{k-1} & \cdots & a_k c_k^{k-1} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = 0$$

By PS10: if all $a_i \neq 0$ (we have this) then this mx is invertible.

Mult on left by inverse: $\begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = 0$

But $v_i \neq 0$. Contradiction.

So: If A $n\times n$,

$\rightarrow$ & $P_A(x)$ has n distinct roots $c_1 \ldots c_n$

then A is diag'ble. $A \sim \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$

What if $P_A(x)$ has $< n$ roots in F?

(i.e. A has $< n$ distinct eigenvalues in F)

$\rightarrow$ Is A diag'ble?

(i.e. $\exists$? basis of eigenvectors)

More complicated.

Ex 1. $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Char poly $P_A(x) = \det(xI - A)$

$= \det\begin{pmatrix} x-2 & 0 \\ 0 & x-2 \end{pmatrix} = (x-2)^2$

Only one root (repeated)

" " eigenvalue ($c=2$)

Every $v \in F^2$ is an eigenvector for A

$T(v) = 2v$ $\quad\quad$ $A = 2I$

Std basis e_1, e_2 is a basis of eigenvectors.

In fact, A is already diagonal.

Ex 2. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Char poly: $P_A(x) = \det\begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$
$= x^2+1.$

a) Say $F = \mathbb{R}$.
x^2+1 has no roots in $\mathbb{R}$

No eigenvalues, no eigenvectors $\neq 0$

Can't be diagonalized (over $\mathbb{R}$)

(ccw rotation by $90°$)

b) $F = \mathbb{C}$.
$x^2+1 = (x+i)(x-i)$
2 distinct roots. $c = i, -i$.
So diagonalizable$/_{\mathbb{C}}$ $A \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

Ex 3. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$P_A(x) = \det\begin{pmatrix} x-1 & -1 \\ 0 & x-1 \end{pmatrix} = (x-1)^2$

One root, $c = 1$. (repeated)

Eigenvectors?

$(A - cI)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$(A - I)\begin{pmatrix} x \\ y \end{pmatrix}.$$

$$A - I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ row reduced.

$y = 0.$ (x arb)

Eigenvectors $(1, 0)$ + multiples.

No basis of eigenvectors.

<u>Not diagble.</u> (over any field)

Criterion? Yes – using min'l poly. (later)

$T : V \longrightarrow V$ lin op

n dim'l.

Basis $\rightsquigarrow A$ $\quad A \sim B$

Another basis $\rightsquigarrow B$

$B = C^{-1} A C$

chg of basis mx.

$P_A(x), \quad P_B(x).$

Relationship?

$$xI - B = xI - C^{-1}AC$$
$$= C^{-1}(xI)C - C^{-1}AC$$
$$= C^{-1}(xI - A)C$$

$xI - B \sim xI - A$

$$P_B(x) = \det(xI - B)$$
$$= \det(xI - A) = P_A(x).$$

So: $P_A(x) = P_B(x)$.

Can refer to $P_T(x)$ (char poly of T)

(any one of these).

Say A is diag'ble.

$$A \sim D = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$$

A, D represent same lin. tr. T

(w.r.t. diff. bases)

$$P_A(x) = P_D(x) = \det(xI - D)$$
$$= \det\begin{pmatrix} x - c_1 & & 0 \\ & \ddots & \\ 0 & & x - c_n \end{pmatrix} = \prod_{i=1}^{n}(x - c_i)$$

Conclusion: If A is diag'ble, then $P_A(x)$ is a product of linear factors $x - c_i$ ← eigenvalues.

(possibly repeated)

List distinct eigenvalues

$\lambda_1, \ldots, \lambda_m$ $\qquad m \le n.$

A arbitrary.

If $P_A(x)$ is a product of lin factors

each is $x - \lambda_i$, ← Occurs d_i times, $d_i \ge 1$

$d_1 + \cdots + d_m = n$

deg $= n$.

$\underset{n \times n}{A} \longleftrightarrow T \qquad P_A(x) = \prod_{i=1}^{m} (x - \lambda_i)^{d_i}$

$\lambda_1, \ldots, \lambda_m \in F$ distinct

Let $W_i = \{$ eigenvectors for $\lambda_i \}$

Claim $W_i \subset V$ is a subspace.

Why? If $v_1, v_2 \in W_i$

$T(v_1) = \lambda_i v_1 \qquad T(v_2) = \lambda_i v_2$

$T(v_1 + v_2) = T(v_1) + T(v_2) = \lambda_i v_1 + \lambda_i v_2 = \lambda_i (v_1 + v_2)$ ✓

$\setminus T(cv_i) = \lambda_i(cv_i)$ Similarly.

Call W_i the eigenspace corresp to λ_i

Say A is diagible. $A \sim D = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$

$v_1 \dots v_n$

$$P_D(x) = P_A(x) = \prod_{i=1}^{m} (x-\lambda_i)^{d_i}$$

$$\overset{\shortparallel}{\prod_{i=1}^{n}(x-c_i)}$$

So: d_1 of the c_i's are $= \lambda_1$

d_2 - - - - - $= \lambda_2$

etc.

Can order $v_1, \dots, v_n$ so that

1^{st} d_1 of them are eigenvectors for λ_1

Next d_2 - - - - - - λ_2

etc.

$v_1, \dots, v_{d_1}$ are all eigenvectors for λ_1

In $\text{span}(v_1, \dots, v_{d_1})$: all are - - -.

$$\underset{W_1}{\cap} \quad \text{dim} = d_1$$

$W_1 \leftarrow$ Can this be bigger?

Ans: **No**.

To see this:

Say $w \in W_1 \subset F^n$

WTS $w \in \mathrm{span}(v_1, \ldots, v_{d_1})$

$v_1, \ldots, v_{d_1}, \ldots v_n$: basis of F^n

$$W_1 \ni w = \underbrace{a_1 v_1 + \cdots + a_{d_1} v_{d_1}}_{w_1 \in W_1} + \underbrace{\quad}_{w_2 \in W_2} \cdots \underbrace{\quad a_n v_n}_{w_m \in W_m}$$

$w = w_1 + w_2 + \cdots + w_m \qquad w_i \in W_i$

$$0 = \underbrace{(w_1 - w)}_{W_1} + \underbrace{w_2}_{W_2} + \cdots + \underbrace{w_m}_{W_m}$$

eigenvectors for (distinct) $\lambda_1 \quad \lambda_2 \cdots \lambda_m$

So $w_1 - w, w_2, \ldots, w_m$: lin. ind. if non-0.

Sum is 0. $\therefore$ they're all 0.

$w_1 = w$

$\quad \|$

$a_1 v_1 + \cdots + a_{d_1} v_{d_1} \in \mathrm{span}(v_1, \ldots v_{d_1})$. ✓

Similarly, same for $W_2 \ldots W_m$.

$A \sim D$ diag.

Conclusion: $W_1 = \text{span}(v_1, \ldots, v_{d_1})$

$W_2 = \text{span}(v_{d_1+1}, \ldots, v_{d_1+d_2})$

etc.

$\dim W_i = d_i$ = # of diag entries equal to λ_i

= exponent of $x - \lambda_i$ in $P_A(x)$.

<u>Prop</u> Say $\dim V = n$ ← vs / F

$T: V \longrightarrow V$, diag'ble,

Say by D, whose distinct diag entries are $\lambda_1, \ldots, \lambda_m$ with λ_i appearing d_i times. (So $n = \sum_{i=1}^{m} d_i$)

Then $P_T(x) = \prod_{i=1}^{m} (x - \lambda_i)^{d_i}$, of degree n, & dim of the eigenspace W_i of λ_i is d_i.

Can get nec. & suff cond'n for diag'bl'ty

Prop V n-dim'l VS / F,
$T: V \to V$, with eigenvalues $\lambda_1, \dots, \lambda_m \in F$
with corresp. eigenspaces $W_1, \dots, W_m \subset V$.
Then TFAE:

i) T is diag'ble, by a diag mx D each of whose diag entries is one of $\lambda_1, \dots, \lambda_m$.

ii) $P_T(x) = \prod_{i=1}^{m} (x-\lambda_i)^{d_i}$ $\Leftarrow$

with $d_i = \dim W_i$.

iii) $\sum_{i=1}^{m} \dim W_i = n$. $\Leftarrow$

Pf. (i) $\Rightarrow$ (ii) by prev prop.

(ii) $\Rightarrow$ (iii) $\sum_{i=1}^{n} \dim W_i = \sum d_i = \deg P_T(x)$
$= n$ ✓

(iii) $\Rightarrow$ (i):

As in prev pf, take a basis of each W_i:

$v_1, \dots, v_{d_1}$ for W_1 $\quad \lambda_1$
$v_{d_1+1}, \dots, v_{d_1+d_2}$ for W_2 $\quad \lambda_2$
$\vdots$ $\quad \vdots$
$v_{\dots}, \dots, v_n$ for W_m $\quad \lambda_m$

<u>Claim:</u> The set $v_1, \dots, v_n$ is lin ind
(& ∴ a basis of V)

Say $0 = \sum a_i v_i$

$$= \underbrace{(a_1 v_1 + \cdots + a_{d_1} v_{d_1})}_{W_1} + \underbrace{(\quad)}_{W_2} + (\quad) \cdots \underbrace{(\quad)}_{W_m}.$$

lin ind of non-0 eigenvectors for distinct eigenvalues

∴ each bracketed term $= 0$.

$$\underbrace{a_1 v_1 + \cdots + a_{d_1} v_{d_1} = 0.}$$ $\quad \dots$

$v_1, \dots, v_{d_1}$: basis of W_1, lin ind

∴ $a_1 = \cdots = a_{d_1} = 0$.

...

∴ all $a_i = 0$. for Claim.

So V has a basis $v_1 \dots v_n$ consisting of eigenvectors for T. ∴ diag'ble

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Back to examples:

<u>Ex 1.</u> $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, diag

$P_A(x) = (x-2)^2$ (ii) holds $\dim W_1 = 2 = \dim V$
$\lambda_1 = 2.$ (iii) holds

Ex 2. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad P_A(x) = x^2+1$

(i) fails.

Over $\mathbb{R}$, not diag'ble.

not a prod of lin factors

No non-0 eigenvectors.

(ii) fails

(iii) fails. (Over $\mathbb{C}$, all hold)

Ex 3 $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, not diag : (i) fails.

One λ, $\lambda = 1$. $P_A(x) = (x-1)^2$

$W_1 = \text{span}(1,0) = x$-axis.

$\dim W_1 = 1. \neq 2$

(ii) fails.

(iii) fails

$1 \neq 2 = \dim$.

What if A (or T) is not diagonalisable?

easy to find det & rank

Can still check if it's Δ'ble.

$$A \sim U = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}, \text{ upper } \Delta\text{r.}$$

(equiv to $\sim$ lower Δr)

reorder basis vectors.

In Ex 3 above: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Not diag'ble. But Δ'r.

$P_A(x) = (x-1)^2$

prod. of lin factors.

In Ex 2 above: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, over $\mathbb{R}$

Not diag'ble. $P_A(x) = x^2+1$

Not a prod. of lin factors/$\mathbb{R}$. No roots in $\mathbb{R}$. No eigenvalues.

Not Δ'ble:

If $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

$A \leftrightarrow T$ $\qquad v_1 \quad v_2$

Then $T(v_1) = a v_1$

non-0 eigenvector.

But T has no such.

Δ'ble $\Downarrow$ $\exists$ eigenvalue.

General Condition:

Prop A is Δ'ble$/_F \Leftrightarrow P_A(x)$ is a product of lin factors over F, $\prod_{i=1}^{n} (x-\lambda_i)^{d_i}$

Here, λ_i's are the eigenvalues. (& diag entries) $\quad (\sum d_i = n)$

Pf. $\Rightarrow$:

$A \sim U$, upper Δr mx

$$U = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$$

$c_1, \ldots c_n$: eigenvalues
Collect like ones
$\lambda_1, \ldots \lambda_m$
$d_1, \ldots d_m$ times

$$P_A(x) = P_U(x) = \det \begin{pmatrix} x-c_1 & & * \\ & \ddots & \\ 0 & & x-c_n \end{pmatrix}$$

$$= \prod_{i=1}^{n} (x - c_i) = \prod_{i=1}^{m} (x-\lambda_i)^{d_i}$$

$$\sum_{i=1}^{m} d_i = n$$

$\Leftarrow$: Use induction on n.

$n=1$: Every 1×1 mx is Δble ✓

Induction step: assume result holds for $n-1$; WTS: holds for n.

$A \in M_n(F)$, suppose $P_A(x)$ = prod. of lin. factors.

WTS: A is Δble $/F$.

Take a linear factor, say $x - c_1$.

c_1 is an eigenvalue, for some non-0 eigenvector $\underset{\neq 0}{v_1}$. $\quad \underline{\underline{T(v_1) = c_1 v_1}}$

Can extend to a basis $v_1, v_2', \ldots, v_n'$ of F^n.

$$A \longleftrightarrow T : F^n \longrightarrow F^n$$

In this new basis, mx for T is

$$A' = \begin{array}{c} \left(\begin{array}{c|c} c_1 & * \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right) \\ v_1 \; v_2' \cdots v_n' \end{array} \qquad B \text{ is } n-1 \times n-1$$

Want to check: $P_B(x)$ is a product of linear factors.

$$A \sim A'$$

$$P_A(x) = P_{A'}(x) = \det\overbrace{(xI - A')}^{\left(\begin{array}{c|c} x-c_1 & * \\ \hline 0 & xI-B \end{array}\right)}$$

expand along 1^{st} column

$$= (x - c_1)\underbrace{\det(xI - B)}_{P_B(x)}$$

$$P_A(x) = (x - c_1) P_B(x).$$

$P_A(x)$ = prod. of linear factors

$(x-c_1)\underbrace{(x-c_2)\cdots(x-c_n)}$

$P_B(x)$ = prod of lin factors.

B is an $n-1 \times n-1$ mx, $\overline{F}$ ...

Ind hyp: B is Δ'ble.

$B \longleftrightarrow S$ span$(v_2', _ v_n')$

lin tr. on a VS of dim $n-1$.

$\exists\, v_2, \ldots, v_n$ of this V.S.

st mx of S is Δ'r.

$$B' = \begin{pmatrix} c_2 & & * \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$$

Now take $v_1, v_2, \_ v_n$

Basis of F^n

Mx of T in this basis is

$$\begin{array}{c|c} c_1 & * \cdots * \\ \hline 0 & \begin{matrix} c_2 & & * \\ & \ddots & \\ 0 & & c_n \end{matrix} \\ \vdots & \\ 0 & \\ v_1 & v_2 \cdots v_n \end{array}$$

: Δ'r.

This case: $F^n = V_1 \oplus V_2$ ← dim $= n-1$

Span(v_1) = V_1 ; dim $= 1$

V_2 = Span$(v_2', \ldots v_n')$ = Span$(v_2 \ldots v_n)$

Another approach:

$$V_1 \subset F^n, \quad W = F^n / V_1$$

$1 \qquad n \qquad n-1$

Work with this

$\mathcal{B}, \mathcal{B}'$

If $F = \mathbb{C}$:

Then: <u>every</u> $n \times n$ mx $A \in M_n(\mathbb{C})$ can be Δd over $\mathbb{C}$.

b/c every $f(x) \in \mathbb{C}[x]$ is a prod. of linear factors

b/c every non-constant $f(x) \in \mathbb{C}[x]$ has a root.

This holds more generally for a field F st every non-const. poly in $F[x]$ has a root in F.

i.e. every mx in $M_n(F)$ can be Δ'd over F.

These fields are called **algebraically closed**.

$\mathbb{C}$ is alg. closed.

$\mathbb{R}$ is not alg closed. (x^2+1)

$\mathbb{F}_2$ - - - - - (x^2-x-1)

$$F = \overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic}\}$$

root of a non-0 poly in $\mathbb{Q}[x]$.

Ex. $\sqrt{2}$ root of x^2-2: alg.

e, π are <u>not alg.</u>

transcendental.

field, alg. closed.

Every field F is contained in an alg. closed field.

— Start with F, adjoin roots of all the poly's $/F$.

Smallest such: algebraic closure of F; $\overline{F}$.

Char poly $P_A(x) = \det(xI - A)$

deg $= n$, monic.

Key property: $P_A(A) = 0$.

i.e. $P_A(x) \in I = \{ f(x) \in F[x] \mid f(A) = 0 \}$.

This shows $I \neq \{0\}$.

We used to show:

$\exists$ monic poly $p_A(x)$, of deg $= d$
$0 < d \leq n$

$p_A(x) \in I$, monic, smallest deg in I,

& $p_A(x) \mid f(x)$ for all $f(x) \in I$.

Still need to show: $P_A(x) \in I$

i.e. $P_A(A) = 0$.

Cayley-Hamilton Thm

H+K, § 6.3, Th 4, pp. 194-196.

A different pf:

1st case: A is Δ'r: $A = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$

$$P_A(x) = \prod_{i=1}^{n} (x - c_i)$$

$P_i(x) = x - c_i$. $\to P_A(x) = \prod_{i=1}^{n} P_i(x)$

$$P_A(A) = \prod_{i=1}^{n} P_i(A) \quad \text{(PS 8 \#5a)}$$

$P_i(A) = A - c_i I$

up Δ'r mx, 0 in (i,i) slot

$= 0$ (by ind on n)

(Ex $n=2$ $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$$\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$) $\therefore$ ✓.

2d case: A is Δ'ble.

$A \sim U$, up Δ'r.

$$P_A(x) = P_U(x)$$

$$p_A(A) = p_U(A) \sim p_U(U) \quad \begin{pmatrix} PS\,6 \\ \#2 \end{pmatrix}$$

$A \sim U$ $\quad \| \quad 0$

By 1st case. ✓

3^{rd} case: A arb.

If F is alg. closed: OK by Case 2.

If F is not alg closed,

$F \subset \underset{\text{alg. cl.}}{E}$ $\qquad$ A is Δble $/E$.

$$p_A(x) = \det(xI - A) \in F[x] \subset E[x]$$

$\underset{0}{\|}$ $\leftarrow$ by working $/E$. ✓

C-H says: $\qquad /F$

$$p_A(A) = 0$$

$p_A(x)$ annihilates A.

$$p_A(x) \in I = \{ f(x) \in F[x] \mid f(A) = 0 \}$$

ideal $\qquad \| \qquad$ {multiples of $p_A(x)$}

$\therefore\ p_A(x) \mid P_A(x)$.

Above pf of C-H uses:

if $A \in M_n(F)$ & $F \underset{\text{fields}}{\subset} E$

then $P_A(x)$ is same $/F$ and $/E$.

$$\begin{array}{c} P_A(x) \\ \| \\ \det(xI - A) \end{array}$$

Q: If A, $E \subset F$ as above, is $p_A(x)$ the same $/F$ and $/E$?

Ans: <u>Yes</u>.

<u>Prop.</u> If $A \in M_n(F)$, & $F \underset{\text{fields}}{\subset} E$, then $p_A(x)$ is the same $/F$ & $/E$.

Pf. Write $p_{A,F}(x)$, $p_{A,E}(x)$ for these min'l polys.

$p_{A,F}(x) \in F[x] \subset E[x]$

$p_{A,F}(A) = 0$. $\quad \therefore\ p_{A,F}(x) \in \underbrace{\{f(x) \in E[x] \mid f(A) = 0\}}_{\text{multiples of } p_{A,E}(x)}$

$\therefore\ p_{A,E}(x) \mid p_{A,F}(x)$

$\uparrow$ this deg $\leq$ $\uparrow$ this degree; & degrees are $=$ iff polys are $=$.

WTS degrees are $=$ (& not $<$)

Let $d = \deg p_{A,E}(x)$.

WTS A satisfies poly of deg d over F.

$p_{A,E}(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d \in E[x]$

$0 = p_{A,E}(A) = a_0 I + a_1 A + \cdots + a_d A^d$

So $I, A, A^2, \ldots, A^d \in M_n(F) \cong F^{n^2}$

are lin. dep. over E; i.e. in

$M_n(E) \cong E^{n^2}$.

By PS7 #2 (for $\mathbb{R} \subset \mathbb{C}$, but same arg works in genl)

$I, A, \ldots, A^d$ lin dep in $M_n(F) \cong F^{n^2}$.

So $\exists\, b_0, \ldots, b_d \in F$, not all 0,

st $\sum b_i A^i = 0$.

A satisfies $f(x) = \sum_{i=0}^{d} b_i x^i$

$\therefore$ min poly $p_{A,F}(x)$ has deg $\leq d$.

Saw: ——— $\geq d$.

$\therefore = d$.

C-H $\Rightarrow p_A(x) \mid P_A(x)$

So every root of $p_A(x)$
is a " " $P_A(x)$ $\leftarrow$ eigenvalues of A $\leftrightarrow$ T

$\therefore$ Every root of $p_A(x)$
is an eigenvalue of A.

Q: Conversely,
= Is every eigenvalue of A
a root of $p_A(x)$?

<u>Ans</u> <u>Yes</u>!

1st a lemma:

<u>Lemma</u> If $T: V \longrightarrow V$ $/F$
has an eigenvector v, with eigenvalue c,
& if $f(x) \in F[x]$ then
$f(T)(v) = f(c)v$.

Pf. $f(x) = \sum b_i x^i \in F[x]$

$f(T)(v) = \sum b_i T^i(v) = \sum b_i c^i v = f(c) v$.

(b/c $T^i(v) = c^i v$) ✓

Now: to show: every eigenvalue of $A^{\alpha T}$ is a root of $p_A(x) = p_T(x)$

If A, B rep. T wrt diff bases, then $A \sim B$. So for $f(x) \in F[x]$,
$f(A) = 0 \Leftrightarrow f(B) = 0$.
$\therefore p_A(x) = p_B(x)$.
So: can refer to $p_T(x)$

For this,

Let c be an eigenvalue of $A^{\alpha T}$ over F.

So: $\exists$ non-0 eigenvector v for c.

$p_A(T) = 0$, $0 = p_A(T)v$

$= \underbrace{p_A(c)}_{\in F} \underset{\neq 0}{v} \Rightarrow p_A(c) = 0.$ ✓

So the roots of $p_A(x)$ are precisely the eigenvalues of A.

So: $p_A(x)$, $P_A(x)$ have the same roots (but poss. with different multiplicities)

Examples — start with easy case — $n \times n$ mx with n distinct eigenvalues

Ex 0. $n=2$. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

eigenvalues: 1, 2.

$P_A(x) = (x-1)(x-2)$ diag.

$\therefore p_A(x) = (x-1)(x-2) = P_A(x)$.

What if don't have n distinct eigenvalues?

Ex 1. $n=2$ $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ diag,

just one eigenvalue (2). $P_A(x) = (x-2)^2$

$p_A(x) = (x-2)$, b/c $A - 2I = 0$.

$\neq P_A(x)$.

Ex 2. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

<u>Over $\mathbb{R}$</u>: no eigenvalues

$P_A(x) = \underbrace{x^2+1}_{\text{irreducible } /\mathbb{R}}$, no root in $\mathbb{R}$.

Not diag'ble, not Δ'ble. $/\mathbb{R}$.

$p_A(x) \mid P_A(x)$ $\therefore p_A(x) = x^2+1 = P_A(x)$.

Over $\mathbb{C}$: $P_A(x) = x^2+1 = (x+i)(x-i)$

2 distinct eigenvalues, $i, -i$. $A \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

diag'ble.

$\therefore\ p_A(x) = (x+i)(x-i) = x^2+1 = P_A(x).$

(same p_A, P_A as $/\mathbb{R}$)

Ex 3. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, only eigenvalue $=1$.

Not diag'ble / any field.

$P_A(x) = (x-1)^2$.

$p_A(x) \mid P_A(x)$, same roots.

either $x-1$ or $(x-1)^2$. $= P_A(x)$

$A - I \neq 0.$

not $p_A(x)$.

$p_A(x) = P_A(x) = (x-1)^2.$

Can use this approach to find $p_A(x)$:

Ex. $A \in M_4(\mathbb{R})$, $P_A(x) = (x-1)(x-2)(x-3)^2$

Then $p_A(x) = (x-1)(x-2)(x-3)$

or $(x-1)(x-2)(x-3)^2$

So: Plug in A into $(x-1)(x-2)(x-3)$.

If get 0, then this is $p_A(x)$.

If not, then $(x-1)(x-2)(x-3)^2$.

Ex 2 over $\mathbb{R}$ is a bit different:

$P_A(x) = x^2+1$ has no roots in $\mathbb{R}$.

$P_A(x)$ is the same $/\mathbb{R}$ + $/\mathbb{C}$.

$\rho_A(x)$ - - - - - - - - - -

Can work $/\mathbb{C}$, & get $\rho_A(x) = x^2+1$

In gen'l, each irred factor of $P_A(x)$ over $\mathbb{R}$ must be an " " " $\rho_A(x)$, over $\mathbb{R}$ — same reason.

More generally: true / any field F:

Reason: pass to a bigger field where the irred factor has roots — e.g. algebraic closure of F $(\overline{F})$

Conclusion: the irred factors $/F$ of $P_A(x)$, & $\rho_A(x)$ are the same. (exc. for multiplicity)

Ex. $A \in M_8(\mathbb{R})$

$$P_A(x) = (x-1)^2(x^2+1)(x^2+2)^2$$

Then

$$p_A(x) = (x-1)^{1 \text{ or } 2}(x^2+1)(x^2+2)^{1 \text{ or } 2}$$

To find $p_A(x)$, start with

$$(x-1)(x^2+1)(x^2+2). \longleftarrow \text{deg } 5$$

If A satisfies this, this is $p_A(x)$.

If not, try an exp of 2 on

$(x-1)$, then – – – – – (x^2+2)
(but not $(x-1)$)

deg 6   deg 7

If these don't work, then $p_A(x) = P_A(x)$.

We saw:

A Δble$/F$ $\iff$ $P_A(x)$ is a prod of lin factors$/F$

$\iff$ $p_A(x)$ – – – – – – – –

<u>Thm</u> (H+K, Th 6, p 204)

A is diag'ble $\iff$

$p_A(x)$ is a prod of <u>distinct</u> linear factors.

Recall: If $P_A(x)$ is a product of distinct linear factors, then A is diag'ble.
— but not conversely.

Let's chk that against the examples:

In Ex 0, Ex 1, Ex 2/$\mathbb{C}$,
$p_A(x)$ is a prod of distinct linear factors & A is diag'ble.

In Ex 2/$\mathbb{R}$, + Ex 3,
$p_A(x)$ is <u>not</u> a prod. of distinct linear factors, and A is <u>not</u> diag'ble.

Re '$\Longrightarrow$' in thm:

let D be an $n \times n$ diag mx.

- diag entries: $\lambda_1, \ldots, \lambda_m$ (distinct)
  multiplicities: $d_1 \ldots d_m$

$$P_D(x) = \prod_{i=1}^{m} (x-\lambda_i)^{d_i}$$

$\Rightarrow p_D(x) = \prod_{i=1}^{m} (x - \lambda_i)$, prod. of distinct lin. factors.

Reason: $D - \lambda_i I$ has a 0 at every diag entry where D has λ_i.

(also 0 off the diag.)

$$\prod_{i=1}^{m} (D - \lambda_i I) = 0$$

i.e. $\prod_{i=1}^{m} (x - \lambda_i)$ becomes 0 if we set $x = D$. ✓

What if A is <u>diag'ble</u>?

Then $A \sim D \leftarrow$ diag.

$$p_A(x) = p_D(x)$$

(b/c if $A \sim B$, then $f(A) \sim f(B)$)

$\therefore p_A(x)$ is a prod of distinct lin. factors. —

This shows '$\Rightarrow$' of Thm.

We'll come back to this.

1st: discussion of invariant subspaces

$T: V \longrightarrow V$ lin tr $/F$

$W \subset V$ subspace.

Can restrict T to W

$T_W : W \longrightarrow V$ restriction of T to W

$T|_W$, $T \upharpoonright W$

(shrink domain)

Image might be smaller.

If $\operatorname{Im} T_W = T(W)$ is contained in W, say

W is <u>invariant under T</u>

(or: T-<u>invariant</u>)

Then: $T_W : W \longrightarrow W$

$$
\begin{tikzcd}
w & V \arrow[r, "T"] & V & T(w) \\
& W \arrow[u, hook] \arrow[r, "T_W"] & W \arrow[u, hook] & \\
& w & & T_W(w) = T(w)
\end{tikzcd}
$$

injective (inclusion)

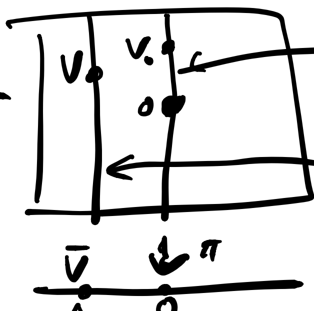
Commutative diagram

$W \subset V$ invariant under $T: V \to V$

$V/W = \{\text{cosets of } W \text{ in } V\}$

quotient space $\qquad v + W$

$v \in V$



$W = y\text{-axis} = \ker \pi$

u

$0 + W$ $\qquad$ 0 elt in V/W

$V = \mathbb{R}^2$

xy plane

$v + W$

$Z = x\text{-axis} \approx V/W$ (1st iso thm)

$(v + W) + (v' + W) = (v + v') + W$

$c(v + W) = cv + W$

$T: V \to V, \qquad W \subset V$

$\quad T$-invariant

We get

$$\overline{T} : V/W \longrightarrow V/W$$
$$v + W \longmapsto T(v) + W$$

Well defined?

What if v, v' are in the same coset?

$v + W = v' + W$ $\qquad$ $v' = v + w$, $\quad w \in W$

$$\begin{array}{ccc} v+W & = & v'+W \\ \overline{T} \downarrow & & \downarrow \\ T(v)+W & \overset{?}{=} & T(v')+W \\ & & \| \\ & & T(v) + \underbrace{T(w)}_{\text{in } W \text{, by invariance}} + W \\ & & \| \\ & & T(v) + W \end{array}$$

✓ =

So get $\overline{T} : V/W \longrightarrow V/W$.

Easy: lin. transf.

Say: $\overline{T}$ is the map on V/W **induced** by T.

Have a comm diagram:

$$\begin{tikzcd}
v \arrow[d, mapsto] & \in V \arrow[r, "T"] \arrow[d, "\text{surj}"'] \arrow[dr, phantom, "///"] & V \arrow[d] \\
v+W & \in V/W \arrow[r, "\overline{T}"] & V/W
\end{tikzcd}$$

$W \subset V \qquad T : V \to V$

invar under T.

$\overline{T} : V/W \to V/W$

If $f(x) \in F[x]$

and $f(T) = 0$.

Then: $f(\overline{T}) = 0. \leftarrow 0+W$

Reason:

$$f(\overline{T})(v+W) = \overbrace{f(T)}^{0}(v) + W$$

$$= 0 + W = W$$

✓ 0-elt in V/W

Take $f = p_T(x)$, get $p_T(\overline{T}) = 0$.

$\therefore\ p_{\overline{T}}(x) \mid p_T(x)$

Use this to prove the "$\Leftarrow$"
of the theorem:

If $p_T(x) =$ prod of distinct linear factors
then T is diag'ble. (on V, n dim'l vs /F)

— — — —.

$p_T(x) = (x-c_1) \cdots (x-c_n)$

$c_1, \ldots, c_n$ distinct eigenvalues of T.

$c_i \leftrightarrow W_i$, eigenspace.

Let $W = $ span of $\bigcup W_i = W_1 \cup \cdots \cup W_n$

$= \cdots \sim$ all eigenvectors

$= W_1 + \cdots + W_n$.

W is invariant under T (PS 11, #3b)

$\therefore$ get $\bar{T} : V/W \longrightarrow V/W$

If we show $W = V$ then

V is spanned by its eigenvectors.

So $\exists$ basis for V consisting of eigenvectors of T.

$\therefore T$ is diag'ble. So — done.

To prove $W = V$, assume not.

If $W \neq V$, $W \subsetneq V$,

then $V/W \neq 0$

$\bar{T} : V/W \longrightarrow V/W$

Min poly $p_{\bar{T}}(x)$ for $\bar{T}$

deg ≥ 1.

$p_{\bar{T}}(x) \mid p_T(x)$, same roots.

prod. of distinct lin factors.

∴ " " " "

" " some of the $x - c_i$'s.

After reordering c_i's,

WMA $(x - c_1) \mid p_{\bar{T}}(x)$.

∴ c_1 is an eigenvalue of $\bar{T}$.

Take a non-0 eigenvector

$\bar{v} = v + W \in V/W$ $\quad v \in V$

for $\bar{T}$, with eigenvalue c_1.

$\bar{v} \neq 0$ $\quad v + W \neq 0 + W = W$

∴ $v \notin W$

$\bar{T}(\bar{v}) = c_1 \bar{v}$. $\quad \bar{v} \neq 0 \in V/W$

If $c' \neq c_1$ then $\bar{T}(\bar{v}) \neq c'\bar{v}$

$c_1, \ldots, c_k$ are distinct. $\quad c_2, \ldots, c_k \neq c_1$.

$$(\bar{T} - c_2 I)\bar{v} = \underbrace{\bar{T}\bar{v}}_{c_1\bar{v}} - c_2\bar{v} \neq 0$$

$$(\underbrace{c_1 - c_2}_{\neq 0})\underbrace{\bar{v}}_{\neq 0} \neq 0$$

Non 0 mult of $\bar{v}$.

Another non 0 eigenvector for $\bar{T}$ with eigenvalue c_1.

Repeat; using $\bar{T} - c_3 I, \ldots, \bar{T} - c_k I$:

Get

$$(\bar{T} - c_k I) \cdots (\bar{T} - c_2 I)\bar{v} = a\bar{v}$$

$$a \neq 0, \quad a = \prod_{i=2}^{k} (c_1 - c_i)$$

Non 0 eigenvector in V/W for $\bar{T}$ with eigenvalue c_1.

$$\underbrace{a\bar{v}}_{V/W} = av + W \neq 0 + W = W$$

$av \notin W$.

$$a = \prod_{i=2}^{h} (c_1 - c_i)$$

$$a\bar{v} = \prod_{i=2}^{h} (T - c_i I)v + W \neq 0 \in V/W$$

$\uparrow$
$\text{``}W$

$$\rightsquigarrow \prod_{i=2}^{h} (T - c_i I)v \notin W.$$

contains all eigenvectors of T

$\therefore$ this is not an eigenvector of T

But: $0 = p_T(T)(v)$

$$= \prod_{i=1}^{h} (T - c_i I)(v)$$

$$= (T - c_1 I) \underbrace{\prod_{i=2}^{h} (T - c_i I)v}_{v'}$$

v' $\therefore$ is an eigenvector for T with eigenvalue c_1.

!

$$\boxed{\begin{aligned} 0 &= (T - c_1 I)v' \\ &= Tv' - c_1 v' \\ \therefore\ Tv' &= c_1 v' \end{aligned}}$$

Contradiction.

This completes the pf of the thm:

Thm (H+K, Th 6, p 204) (diff. pf.)

$T \leftrightarrow A$ is diag'ble $\iff$ $p_A(x)$ ($\text{``}p_T(x)$) is a prod. of distinct lin. factors.

To use this to determine diagonalizability (over F)

$A \xrightarrow[\text{find}]{} P_A(x)$.

We could then find $p_A(x)$:

Factor $P_A(x)$ over F,

test products of irred factors taking each at least once.

Can do even less!

Factor $P_A(x)$ over F; look at irred factors.

If __any__ irred factor of $P_A(x)$ is non-linear, then A is not diag'ble.

If __all__ of the irred factors of $P_A(x)$ are linear, then:

$$P_A(x) = \prod_{i=1}^{k} (x-\lambda_i)^{d_i}$$

$\lambda_1, \ldots, \lambda_k$ distinct.

Test $f(x) = \prod_{i=1}^{k} (x - \lambda_i)$:

If $f(A) = 0$ then $f(x) = \mu_A(x)$,
& A is diag'ble.

If $f(A) \neq 0$, then $\mu_A(x)$ has a factor with an exponent > 1,
so not a product of distinct linear factors,
& so A is not diag'ble.

Add'l results about diag'n & Δ'n:

Say have a collection of lin transf's
$$T_i : V \longrightarrow V$$
f.d.v.s $/F$

(or corresp m'x's A_i)

Suppose that each is Δble.
(i.e. $\forall i$ $\exists$ basis $\mathcal{B}_i$ of V making T_i (A_i) Δ'r)

Q: Is there a single basis $\mathcal{B}$ that makes them all Δ'r.
(Simultaneous Δ'n)

In gen'l: **No.** But:

Thm (Th 7, H&K, p207):

If T_i's (or A_i's)

commute with each other

$(\forall i,j \quad T_i T_j = T_j T_i)$

then yes.

The pf is related to the pf

that lin tr. is Δble $\Leftrightarrow$ min poly

is a prod of lin. factors.

Cor (Cor to Th 7 above):

If F is alg. closed (eg. $F = \mathbb{C}$),

if $T_i : V \longrightarrow V$ commute, $\leftrightarrow A_i$

then: $\exists\, C$, invertible, st

$C^{-1} A_i C$ is upper Δ'r $\forall i$.

Also: analog of result for diag'lity:

f.d.v.s / F

(Th 8, H&K, p207

If T_i are a family of diag'ble commuting lin. tr.: $V \to V$

then $\exists$ b.... $\mathcal{B}$ making all T_i's diag.

Simultaneous diagn.

Pf uses similar ideas, to prev. thm.

Can phrase in terms of mx's A_i instead of lin tr's T_i.