Math 314 Problem Set #5 Due week of Feb. 18, 2019, in lab.

Reminder: Exam #1 will take place in class on Monday, February 18, on the material covered in class up to that point, and on homeworks up through PS 5. Sample Exam 1 is available on the course web page. Those who submit the sample exam to their TA by 3pm on Friday, Feb. 15, or who submit it in lecture on Feb. 15, will get extra credit. There are also more study problems for Exam 1 on the course web page (as an aid to study, not for additional extra credit). During the actual exam, no electronic devices, notes, or other aids can be used, except for a single two-sided handwritten 5"x7" index card.

Read Hoffman and Kunze, Chapter 3, Sections 2 and 3.

1. For each of the following, either give an example or explain why no such example exists:
   a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose kernel is one-dimensional.
   b) A linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ whose kernel is trivial.
   c) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ taking $(1, 2, 3)$ to $(4, 5)$.

2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.
   a) Show that the kernel of T is contained in the kernel of $T^2 = T \circ T$. Can the two kernels ever be equal? Can they ever be unequal? Explain (with examples).
   b) Show that $\text{image}(T) \subset \text{kernel}(T) \Leftrightarrow T^2 = 0$.

3. Let $T : V \to W$ and $S : W \to X$ be linear transformations of vector spaces. What inequalities (i.e. $\leq$) can you find relating these quantities: $\text{rank}(T)$, $\text{rank}(S)$, $\text{rank}(S \circ T)$, $\dim(V)$, $\dim(W)$, $\dim(X)$? (For example, can you relate $\text{rank}(S \circ T)$ to all the other quantities?)

4. Let A be a 2×3 matrix. Show that there cannot be any 3×2 matrix B such that BA is the 3×3 identity matrix. But also show that (depending on A) it may be possible to find a 3×2 matrix B such that AB is the 2×2 identity matrix. (Hint: Problem 3 may help.)

5. Let $W \subset \mathbb{R}^3$ be the subspace given by $x + y + z = 0$. Find a basis of W and extend it to a basis of $\mathbb{R}^3$. Find another basis of $\mathbb{R}^3$ that does not contain *any* vector in W.