Math 314 Problem Set #7 Due week of March 11, 2019, in lab.

In Hoffman and Kunze, read Chapter 3, Sections 5 and 6; and Appendix A.4.

1. From Hoffman and Kunze, Chapter 3, do these problems: pages 105-107, #1-5; page 111, #1.

2. Let m, n be positive integers and consider vectors $v_1, \dots, v_m \in \mathbb{R}^n$. Show that these vectors are linearly independent as vectors in $\mathbb{R}^n$ over the field of scalars $\mathbb{R}$ if and only if they are linearly independent as vectors in $\mathbb{C}^n$ over the field of scalars $\mathbb{C}$. (Hint: One direction is easy. For the other direction, consider the following question: If A is an $m \times n$ matrix with reduced row echelon form R, then how can one tell from R whether the rows of A are linearly independent?)

3. Let V be a vector space, let W be a subspace of V, and let S be a subset of V.

a) If S is a linearly independent subset of V, must $S \cap W$ be a linearly indendent subset of W?

b) If S spans V, must $S \cap W$ span W?

c) If S is a basis of V, must $S \cap W$ be a basis of W?

d) If $\mathrm{ann}(W) \subset \mathrm{ann}(S)$, must $S \subset W$?

e) If $\mathrm{ann}(S) \subset \mathrm{ann}(W)$, must $W \subset S$?

4. If X, Y are subspaces of a vector space V, write $V = X \oplus Y$ if every element $v \in V$ can be written *in exactly one way* as $v = x + y$ with $x \in X$ and $y \in Y$.

a) If $V = \mathbb{R}^3$ and X is the x-axis, find a subspace $Y \subset V$ such that $V = X \oplus Y$. Find the dimensions of V, X, Y, V^*, $\mathrm{ann}(X)$, $\mathrm{ann}(Y)$. What relationships do you notice among these dimensions?

b) Let V be any finite dimensional vector space with subspaces X, Y. Show that $V = X \oplus Y$ if and only if the following two conditions both hold: $X + Y = V$ and $X \cap Y = 0$.

c) Let V be a finite dimensional vector space with subspaces X, Y, such that $V = X \oplus Y$.

i) Show that if $\mathcal{A}$ is a basis of X and $\mathcal{B}$ is a basis of Y, then $\mathcal{A} \cup \mathcal{B}$ is a basis of V.

ii) Prove that the numerical relationships you noticed in part (a) hold.

iii) Show that $V^* = \mathrm{ann}(X) \oplus \mathrm{ann}(Y)$.

5. For any finite dimensional vector space V with basis $\mathcal{B} = \{v_1, \dots, v_n\}$, and corresponding dual basis $\mathcal{B}^* = \{f_1, \dots, f_n\}$ of V^*, define $\phi_{V,\mathcal{B}} : V \to V^*$ by $\sum_1^n a_i v_i \mapsto \sum_1^n a_i f_i$. In particular, since V^* is finite dimensional with basis $\mathcal{B}^*$, we can also consider the map $\phi_{V^*,\mathcal{B}^*} : V^* \to V^{**}$. Let $\psi_{V,\mathcal{B}} = \phi_{V^*,\mathcal{B}^*} \circ \phi_{V,\mathcal{B}} : V \to V^{**}$.

a) Show that $\phi_{V,\mathcal{B}} : V \to V^*$ is an isomorphism, but that it depends on the choice of basis $\mathcal{B}$. [Hint: For the second part, choose two different bases $\mathcal{B}, \mathcal{B}'$ of some vector space V; e.g. take V to be the one-dimensional space $\mathbb{R}$. Then compare the two maps $\phi_{V,\mathcal{B}}$ and $\phi_{V,\mathcal{B}'}$, and verify that they are not the same.]

b) Explain what $\psi_{V,\mathcal{B}}$ does to each basis vector of V, and show that $\psi_{V,\mathcal{B}} : V \to V^{**}$ is an isomorphism. Also show that $\psi_{V,\mathcal{B}}$ is the same as the isomorphism $\mathrm{ev} : V \to V^{**}$ given by $v \to \mathrm{ev}_v$, where $\mathrm{ev}_v(f) = f(v)$ for $f \in V^*$. (Hint: Show $\psi_{V,\mathcal{B}}(v_i) = \mathrm{ev}_{v_i}$ for all i.) Then deduce that $\psi_{V,\mathcal{B}}$ does *not* depend on the choice of basis $\mathcal{B}$ (and in that sense is "natural").