Math 314 Problem Set #11 Due week of April 8, 2019, in lab.

Read Hoffman and Kunze, Chapter 6, Sections 3-4.

1. From Hoffman and Kunze, Chapter 6, do these problems: Pages 197-198, #1,2,5,6,7,8; pages 205-206, #5,7,10.

2. For each of the following matrices, find the characteristic polynomial and the minimal polynomial; determine whether the matrix is similar to a real diagonal matrix and whether it is similar to a real triangular matrix; also whether it is similar to a complex diagonal or triangular matrix. In each case find the diagonal matrix it is similar to, if it exists.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

3. Let $T : V \to V$ be a linear transformation, and let $S$ be the set of eigenvectors of $T$ (including 0).

a) Show that $S$ is *not* always a subspace of $V$. Can $S$ *ever* be a subspace of $V$? How does this relate to eigenspaces, and to problem 6 on Problem Set 2?

b) Let $W$ be the span of $S$. Show that $W$ is *invariant* under $T$, i.e. $T(W) \subset W$. Must $W = V$?

4. Let $A$ be an $n \times n$ matrix over $\mathbb{R}$ that has $n$ distinct eigenvalues.

a) Show that there is a cube root of $A$; i.e. a matrix $B \in M_n(\mathbb{R})$ such that $B^3 = A$.

b) Find necessary and sufficient conditions on the eigenvalues in order for $A$ to have a square root in $M_n(\mathbb{R})$.

c) Determine whether the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \in M_2(\mathbb{R})$$

has a square root in $M_2(\mathbb{R})$, and whether it has a cube root in $M_2(\mathbb{R})$. In each case, if such a root exists, find one.

5. a) Given a complex matrix $A$ and a sequence of complex matrices $A_1, A_2, A_3, \ldots$, we say that $\lim_{i\to\infty} A_i = A$ if each entry of $A$ is the limit of the corresponding entries of the matrices $A_i$. Show that every matrix $A$ in $M_2(\mathbb{C})$ is the limit of some sequence of diagonalizable matrices $A_i$ in $M_2(\mathbb{C})$. [Hint: An $n \times n$ matrix is diagonalizable if it has $n$ distinct eigenvalues.]

b) Use this to give a different proof of the Cayley-Hamilton theorem for $M_2(\mathbb{C})$; and then for $M_2(\mathbb{R})$. [Hint: For the first part, observe that Cayley-Hamilton holds for diagonal matrices; and for the second part, use that the characteristic polynomial doesn't change upon field extension.]

c) [Optional] Generalize parts (a) and (b) to $M_n(\mathbb{C})$ and $M_n(\mathbb{R})$.