Instructions: These are additional study problems for Exam #2. To practice for the exam, do all of these problems, showing your work and explaining your assertions. Allow yourself 10 minutes per problem. These problems supplement the problems on Sample Exam #2.

1. Let A, B, C be $n \times n$ matrices over a field F, with C invertible and $B = C^{-1}AC$. Show that A and B have the same rank and have the same nullity.

2. Let $T : V \to W$ and $S : W \to V$ be linear transformations of finite dimensional vector spaces, such that $S \circ T$ is the identity on V. Must T be an isomorphism of V with W? Either show that this is the case, or give a counterexample.

3. Let $V = \mathbb{R}^3$, with basis i, j, k, and let x, y, z be the dual basis of V^*. Find a basis for the annihilator of the subspace of V spanned by $3i - j$.

4. Show that if V is a finite dimensional vector space, and $v, w \in V$ are linearly independent, then there exists $f \in V^*$ such that $f(v) = 1$ and $f(w) = 2$. What if v, w are instead linearly dependent?

5. Let $V = \mathbb{R}^4$ with coordinates $x_1, \ldots, x_4$, and let W be the subspace of V given by $x_1 + \cdots + x_4 = 0$. Let $T : V \to V/W$ be the linear transformation taking $v \in V$ to the coset $v + W \in V/W$. Find a subspace $X \subset V$ such that the restriction of T to X is an isomorphism $X \to V/W$, and show that $V = W \oplus X$.

6. Let $T : V \to W$ be a linear transformation, and let $T^* : W^* \to V^*$ be its transpose. Show that $\ker(T^*) = \mathrm{ann}(\mathrm{im}(T))$.

7. Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 4 & 0 \\ 2 & 0 & -5 & 0 \end{pmatrix}$$

and determine if this matrix has an inverse. [Hint: Some rows or columns may be easier than others to expand about.] Does your answer depend on the field of scalars?