Math 370 Exam #2 solutions Nov., 2007

1. We have shown that if a matrix A represents a linear transformation T with respect to some basis, then the rank of T equals the rank of A. Since $B = C^{-1}AC$, the matrices A and B represent the same linear transformation T with respect to different bases. So $\text{rank}(A) = \text{rank}(T) = \text{rank}(B)$. Since $\text{rank}(A) + \text{nullity}(A) = n$ and similarly for B, we also have that $\text{nullity}(A) = \text{nullity}(B)$.

2. Let S be the span of $3i - j$ in V. An element $f \in V^*$ is of the form $ax + by + cz$ with $a, b, c \in F$. Here $f \in \text{Ann}(S)$ if and only if $0 = f(3i - j) = 3a - b$. The set of f's with $3a - b = 0$ has basis $\{x + 3y, z\}$.

3. ϕ is a linear transformation because $\phi(f+g) = (x^2+1)(f+g) = (x^2+1)f + (x^2+1)g = \phi(f) + \phi(g)$ and because $\phi(cf) = (x^2+1)cf = c\phi(f)$. ϕ is not an algebra homomorphism because $\phi(1 \cdot 1) = \phi(1) = x^2 + 1 \neq (x^2+1)^2 = \phi(1) \cdot \phi(1)$. ψ is not a linear transformation because $\psi(2x) = (2x)^2 = 4x^2 \neq 2\psi(x)$. So ψ is also not an algebra homomorphism.

4. $f \in V$ is in $\ker(T)$ if and only if f vanishes at $x = 1, 2, 3, 4$; or equivalently, if and only if $f(x)$ is divisible by $x-1, x-2, x-3, x-4$. Since those four polynomials are irreducible, by unique factorization the above condition is equivalent to saying that f is divisible by the product $(x-1)(x-2)(x-3)(x-4)$. So $\ker(T)$ is the subset of V consisting of polynomial multiples of $(x-1)(x-2)(x-3)(x-4)$. But elements of V have degree at most 4. So the kernel consists of the set of *scalar* multiples of $(x-1)(x-2)(x-3)(x-4)$; this is a one-dimensional subspace spanned by $(x-1)(x-2)(x-3)(x-4)$. Since $\text{rank}(T) + \text{nullity}(T) = \dim(V) = 5$, it follows that $\text{rank}(T) = 4$ and so the image of T is all of $\mathbb{R}^4$.

5. The polynomials $(x^3+x^2)a(x) + (x^2-1)b(x)$ form a principal ideal I in $\mathbb{R}[x]$, generated by the g.c.d. of $(x^3 + x^2)$ and $(x^2 - 1)$. So the monic element of least degree is the g.c.d., which is $x + 1$ because $x^3 + x^2 = x^2(x+1)$ and $x^2 - 1 = (x-1)(x+1)$. This element is unique because if g is another choice, then f divides g since $g \in I$, and then $f = g$ since f, g are monic of the same degree.