Read Hoffman and Kunze, Chapter 3, Section 6.

1. From Hoffman and Kunze, Chapter 3, do these problems:
Pages 105-107, #4, 5, 11, 12, 17.

2. a) Let V, W, X be finite dimensional vector spaces, and let $S : W \to X$ and $T : V \to W$ be linear transformations. Show that $\text{rank}(S \circ T) \le \text{rank}(T)$ and that $\text{rank}(S \circ T) \le \text{rank}(S)$.

b) Let $T : V \to V$ be a linear transformation, where V is a finite dimensional vector space. Let T^n denote $T \circ T \circ \cdots \circ T$ (with n factors of T) and let $r_n = \text{rank}(T_n)$. Prove that $r_{n+1} \le r_n$ for all n, and deduce that the sequence $r_1, r_2, r_3, \ldots$ is eventually constant.

3. Let $\mathcal{P}_n$ be the vector space of real polynomials $f(x)$ of degree at most n. Define $T : \mathcal{P}_2 \to \mathcal{P}_2$ by $f(x) \mapsto f(x) - (x-1)f'(x)$, where $f'(x)$ is the derivative of $f(x)$.

a) Show that T is a linear transformation.

b) Find the kernel of T. [Hint: Differential equations.]

c) Find the matrix of T with respect to the basis $\{1, x, x^2\}$ of $\mathcal{P}_2$.

d) Find the matrix of T with respect to the basis $\{f_1, f_2, f_3\}$ of $\mathcal{P}_2$, where $f_i = F^{-1}(e_i)$ as in problem 4 of Problem Set #6.

4. For any finite dimensional vector space V with basis $B = \{e_1, \ldots, e_n\}$, and corresponding dual basis $B^* = \{\delta_1, \ldots, \delta_n\}$ of V^*, define $\phi_{V,B} : V \to V^*$ by $\sum_1^n a_i e_i \mapsto \sum_1^n a_i \delta_i$. Also let $\psi_{V,B} = \phi_{V^*,B^*} \circ \phi_{V,B} : V \to V^{**}$.

a) Show that $\phi_{V,B} : V \to V^*$ is an isomorphism, but that it depends on the choice of basis B. [Hint: For the second part, choose two different bases of some vector space V; e.g. take V to be the one-dimensional space $\mathbb{R}$.]

b) Explain what $\psi_{V,B}$ does to each basis vector of V, and show that $\psi_{V,B} : V \to V^{**}$ is an isomorphism. Also show that $\psi_{V,B}$ is the same as the isomorphism $\text{ev} : V \to V^{**}$ given by $v \to \text{ev}_v$, where $\text{ev}_v(f) = f(v)$ for $f \in V^*$. [Hint: Show $\psi_{V,B}(e_i) = \text{ev}_{e_i}$ for all i.] Deduce that $\psi_{V,B}$ does *not* depend on the choice of basis B (and in that sense is "natural").