Math 370 Problem Set #9 Due week of Nov. 5, 2007, in lab.

Read Hoffman and Kunze, Chapter 4, Section 4.

1. From Hoffman and Kunze, Chapter 4, do these problems:
Page 123, #7, 8. Page 126, #1. Page 134, #1, 3.

2. Prove or disprove each of the following assertions:

a) If V is a finite dimensional vector space with basis $B = \{v_1, \ldots, v_n\}$, and W is a subspace of V, then $B \cap W$ is a basis for W.

b) If V is a vector space, and S is a linearly independent subset of V that is not contained in *any* strictly larger linearly independent subset of V, then S is a basis of V.

3. Let $\mathcal{A}$ be an algebra over a field F.

a) Show that $0 \cdot a = 0$ for all $a \in \mathcal{A}$, where $0 \in \mathcal{A}$ is the additive identity in $\mathcal{A}$.

b) Suppose that $\mathcal{A}$ has a multiplicative identity $1 \in \mathcal{A}$. Prove that $(-1) \cdot a = -a$ for all $a \in \mathcal{A}$ (where $-a \in \mathcal{A}$ denotes the additive inverse of $a \in \mathcal{A}$).

4. a) Let F be a field, let $f(x) \in F[x]$, and let A be an $n \times n$ matrix over F. Suppose that $f(x) = f_1(x)f_2(x)$ in $F[x]$. Prove that $f(A) = f_1(A)f_2(A)$ as matrices.

b) Let $f(x, y) \in F[x, y]$, the algebra of polynomials in x and y with coefficients in F. Let A, B be $n \times n$ matrices over F. Suppose that $f(x, y) = f_1(x, y)f_2(x, y)$ in $F[x, y]$. Show that $f(A, B)$ is *not* necessarily equal to the matrix $f_1(A, B)f_2(A, B)$. [Hint: Let $f(x, y) = x^2 - y^2$ and pick two 2×2 matrices.]

c) Explain where your proof for (a) breaks down in (b).

5. Show that the division algorithm for polynomials (Hoffman & Kunze, §4.4, Theorem 4) does not remain true if the coefficient field F is replaced by $\mathbb{Z}$.

6. Which of the following are homomorphisms of $\mathbb{R}$-algebras? For those that are not, why not? For those that are, what are the kernels and images?

a) $f : \mathbb{R}[x] \to \mathbb{R}$, $f(\sum a_i x^i) = \sum a_i 3^i \quad (a_i \in \mathbb{R})$

b) $f : \mathbb{C} \to \mathbb{C}$, $f(a + bi) = a - bi \quad (a, b \in \mathbb{R})$

c) $f : \mathbb{C} \to \mathbb{C}$, $f(a + bi) = a \quad (a, b \in \mathbb{R})$