Math 370 Problem Set #10 Due week of Nov. 12, 2007, in lab.

Reminder: Exam #2 will take place in class on Monday, November 12, on the material covered in class since the previous exam, and up through the end of Chapter 4 of the text.

Read Hoffman and Kunze, Chapter 4, Section 5 and Chapter 5, Section 1.

1. From Hoffman and Kunze, Chapter 4, do these problems:
Page 134, #2(b). Page 139, #1, 5-8. [In problems 5-8, use the definition given just before problem 5. In problem 7, use problem 6.]

2. Let $a(x), b(x), c(x) \in F[x]$, where F is a field. Suppose that $a(x)$ and $b(x)$ are relatively prime, and also that $a(x)$ divides $b(x)c(x)$. Show that $a(x)$ divides $c(x)$. [Hint: Do this analogously to the proof of Theorem 8 of Hoffman and Kunze, §4.5.]

3. Let $a(x), b(x) \in F[x]$, where F is a field. Let d be the largest degree of any polynomial that divides both $a(x)$ and $b(x)$.

a) Show that there is a unique monic polynomial $g(x)$ of degree d such that $g(x)$ divides both $a(x)$ and $b(x)$.

b) Show that this polynomial $g(x)$ is the greatest common divisor of $a(x)$ and $b(x)$; i.e. $g(x)$ is the monic generator of the ideal $(a(x), b(x))$.

4. a) Let $f(x), g(x) \in F[x]$, where F is a field. Suppose that the only polynomials in $F[x]$ that divide both $f(x)$ and $g(x)$ are constant polynomials. Show that $\{af + bg \mid a, b \in F[x]\}$ is all of $F[x]$.

b) Show that if $F[x]$ is replaced by $F[x, y]$, then the conclusion in (a) is no longer true.

5. a) Find a surjective homomorphism of $\mathbb{R}$-algebras $\mathbb{R}[x] \to \mathcal{B}$ whose kernel is the ideal generated by $x + 1$.

b) Do the same with $x + 1$ replaced by $x^2 + 1$.