

Review Hoffman and Kunze, Chapter 6, Sections 1-2.

1. From Hoffman and Kunze, Chapter 6, do these problems:

Page 189, #1-5, 10.

2. Consider the system of linear equations  $AX = B$ , where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -3 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

Solve this system in three different ways:

a) By row reduction on the augmented matrix  $(A | B)$ .

b) By finding  $A^{-1}$  and writing  $X = A^{-1}B$ .

c) By Cramer's Rule.

3. Let  $c_1, \dots, c_k \in F$  be distinct elements in a field  $F$ .

a) Show that there is no non-zero vector  $(b_0, \dots, b_{k-1}) \in F^k$  such that  $\sum_{j=0}^{k-1} b_j c_i^j = 0$

for all  $i = 1, \dots, k$ . [Hint: Consider roots of the polynomial  $f(x) = \sum_{j=0}^{k-1} b_j x^j$ .]

b) Let  $a_1, \dots, a_k \in F$ . Using part (a), show that the matrix

$$\begin{pmatrix} a_1 & a_2 & \dots & a_k \\ a_1 c_1 & a_2 c_2 & \dots & a_k c_k \\ a_1 c_1^2 & a_2 c_2^2 & \dots & a_k c_k^2 \\ \vdots & \vdots & \dots & \vdots \\ a_1 c_1^{k-1} & a_2 c_2^{k-1} & \dots & a_k c_k^{k-1} \end{pmatrix}$$

is invertible, unless some  $a_i = 0$ .

4. Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  take  $(a, b) \mapsto (17a - 30b, 9a - 16b)$ .

a) Find the matrix of  $T$  with respect to the standard basis  $e_1, e_2$ .

b) Find the matrix of  $T$  with respect to the basis  $f_1 = (1, 1), f_2 = (1, -1)$ .

c) Find a basis of  $\mathbb{R}^2$  for which the matrix of  $T$  is the diagonal matrix  $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ .

[Hint: Eigenvalues, eigenvectors.]

d) Is there a basis of  $\mathbb{R}^2$  for which  $T$  has matrix  $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ ? Explain.

5. Let  $T : V \rightarrow V$  be a linear transformation, and let  $S$  be the set of eigenvectors of  $T$  together with 0.

a) Show that if  $c$  is an eigenvalue of  $T$ , then the eigenspace  $W_c = \{v \in V \mid T(v) = cv\}$  is really a subspace of  $V$ .

b) Show by example that  $S$  is *not* always a subspace of  $V$ . Can it *ever* be a subspace of  $V$ ?

c) Let  $W$  be the span of  $S$ . Show that  $W$  is *invariant* under  $T$ , i.e.  $T(W) \subset W$ .