Math 370 Problem Set #13 Due week of Dec. 3, 2007, in lab.

Read Hoffman and Kunze, Chapter 6, Section 3, and Chapter 8, Sections 1 and 2.

1. From Hoffman and Kunze, Chapter 6, do these problems:
Pages 197-198, #1-6.

2. From Hoffman and Kunze, Chapter 8, do these problems:
Pages 275-276, #1, 5, 9, 12. [In #5, $\{\epsilon_1, \epsilon_2\}$ denotes the standard basis.]
[Hint for #12: Define $T : V \to \mathbb{R}^n$ by $\alpha \mapsto ((\alpha|\alpha_1), \dots, (\alpha|\alpha_n))$. Is $T$ an isomorphism?]

3. Let $\mathcal{P}_2$ be the real vector space of polynomials in $\mathbb{R}[x]$ of degree at most 2. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear transformation given by $T(f) = g$ where $g(x) = (x+1)f'(x)$. Find the eigenvalues and eigenvectors of $T$. Do this in *two different* ways, namely:

i) Find the matrix of $T$ relative to the basis $\{1, x, x^2\}$ and use that.

ii) Instead, use separation of variables to solve the differential equation $(x+1)\cdot\frac{dy}{dx} = cy$, where $c$ is a constant.

4. For each of the following matrices, find the characteristic polynomial and the minimal polynomial; determine whether the matrix is similar to a real diagonal matrix and whether it is similar to a real triangular matrix; also whether it is similar to a complex diagonal or triangular matrix. In each case find the diagonal matrix it is similar to, if it exists.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

5. a) Show that if $T : V \to V$ is a linear transformation, and $v \in V$ is an eigenvector for $T$ with eigenvalue $c$, then $v$ is also an eigenvector for $T^k$, with eigenvalue $c^k$.

b) Use this to find the eigenvalues and corresponding eigenvectors of $A^{253}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix}.$$

[Note: You are not required to find the entries of the matrix $A^{253}$.]

6. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Show that $T$ *preserves length* (i.e. $\|T(x)\| = \|x\|$ for all $x \in \mathbb{R}^n$) if and only if $T$ *preserves inner product* (i.e. $T(x)\cdot T(y) = x \cdot y$ for all $x, y \in \mathbb{R}^n$). [Hint: $\|x\|^2 = x \cdot x$, so $\|T(x+y)\|^2 = \cdots$.]