Math 502 Problem Set #8 Due the week of Oct. 30, 2006, in lab.

Review Artin, Chapter 4, sections 1-3.

From Artin, do these problems:

Section 4.1 (p.145): 2, 9. [The notation $W < V$ or $V > W$ means that W is a subspace of V such that $W \neq V$.]

Section 4.3 (pp.146-147): 3, 6, 9, 10.

[Optional, for extra credit] Miscellaneous problems for Chapter 3 (p.108): 8.

Also do the following problems:

1. Show that no two of these groups of order 8 are isomorphic: the quaternion group, the dihedral group D_4, the cyclic group C_8, and the product $C_4 \times C_2$.

2. Let G be a non-trivial subgroup of the additive group $\mathbb{C}$. Suppose that G is *discrete* (i.e. there exists $r > 0$ such that the distance between any two distinct elements of G is at least r).

a) Give an example of such a G that is generated by one element.

b) Give an example of such a G that is generated by a set of two elements, but which cannot be generated by just one element.

c) Show that G has a non-zero element v closest to 0 (i.e. such that no non-zero element of G is closer to 0). [Hint: Use discreteness.]

d) Show that G has a generating set consisting of at most two elements. [Hint: Let v be as in (c); and let $w \in G$ be an element *not* in $\langle v \rangle$ that is closest to 0 (among those not in $\langle v \rangle$). Show $G = \langle v, w \rangle$.]

3. Let F be a field. For each of the following, either give an example or explain why no such example exists:

a) A linear transformation $F^2 \to F^2$ whose kernel is one-dimensional.

b) A linear transformation $F^4 \to F^3$ whose kernel is trivial.

c) A linear transformation $F^3 \to F^2$ taking $(1, 2, 3)$ to $(4, 5)$.

4. Suppose that A and B are similar $n \times n$ matrices over a field F (i.e. $B = C^{-1}AC$ for some $C \in \mathrm{GL}_n(F)$).

a) Show that A^n and B^n are similar.

b) Show that if $f(x) \in F[x]$, then $f(A)$ and $f(B)$ are similar.

c) Show that if $f(x) \in F[x]$ and $f(A) = 0$, then $f(B) = 0$.

5. a) Let V, W, X be finite dimensional vector spaces, and let $S : W \to X$ and $T : V \to W$ be linear transformations. Show that $\mathrm{rank}(S \circ T) \leq \mathrm{rank}(T)$ and that $\mathrm{rank}(S \circ T) \leq \mathrm{rank}(S)$.

b) Let $T : V \to V$ be a linear operator on a finite dimensional vector space V. Let r_n be the rank of $T^n = T \circ T \circ \cdots \circ T$ (with n copies of T). Prove that $r_{n+1} \leq r_n$ for all n, and deduce that the sequence $r_1, r_2, r_3, \ldots$ is eventually constant.

c) What happens in (b) if instead V is infinite dimensional?

6. a) Find polynomials f_0, f_1, f_2 of degree ≤ 2 in $\mathbb{R}[x]$ such that for all $0 \leq i, j \leq 2$ we have that $f_i(j) = 1$ if $i = j$ and $f_i(j) = 0$ if $i \neq j$.

b) Prove that f_0, f_1, f_2 form a basis of $P_2 = \{\text{polynomials of degree } \leq 2\}$.