Math 502 Problem Set #9 Due the week of Nov. 6, 2006, in lab.

Read Artin, Chapter 4, sections 4 and 6.

From Artin, do these problems:

Section 4.4 (pp.147-148): 1, 2, 3, 7, 9. In #3, say what happens for $\theta = \pi/2$.

Section 4.6 (pp.149-150): 1, 2, 4, 7, 10. In #2, say what happens for $\theta = \pi/2$.

[Optional, for extra credit] Miscellaneous problems (pp.152-154): 8.

Also do the following problems:

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ take $(a,b) \mapsto (17a - 30b, 9a - 16b)$.

a) Find the matrix of T with respect to the basis $e_1 = (1,0)$, $e_2 = (0,1)$.

b) Find the matrix of T with respect to the basis $f_1 = (1,1)$, $f_2 = (1,-1)$.

c) Find a basis of $\mathbb{R}^2$ for which the matrix of T is the diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

d) Is there a basis of $\mathbb{R}^2$ for which T has matrix $\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$? Explain.

2. Let $T : V \to V$ be a linear operator, and let S be the set of eigenvectors of T together with 0.

a) Show that if c is an eigenvalue of T, then the eigenspace $W_c = \{v \in V \mid T(v) = cv\}$ is really a subspace of V.

b) Show by example that S is *not* always a subspace of V. Can it *ever* be a subspace?

c) Let W be the span of S. Show that W is T-invariant, i.e. $T(W) \subset W$.

3. Let P_2 be the real vector space of polynomials in $\mathbb{R}[x]$ having degree ≤ 2. Let T be the linear operator on P_2 defined by $T(f) = g$, where $g(x) = (x+1)f'(x)$. Find the eigenvalues and eigenvectors of T. Do this in *two different ways*, namely:

i) Find the matrix of T relative to the basis $\{1, x, x^2\}$, and use that.

ii) Instead, use separation of variables to solve the differential equation $(x+1)\cdot\frac{dy}{dx} = cy$, where $c \in \mathbb{R}$.

4. For each of the following matrices, find the characteristic polynomial and the minimal polynomial; determine whether the matrix is similar to a real diagonal matrix and whether it is similar to a real triangular matrix; also whether it is similar to a complex diagonal or triangular matrix. In each case, find the diagonal matrix it is similar to, if possible:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$