Math 503 Problem Set #3 Due the week of Jan. 29, 2007, in lab.

Read Artin, Chapter 10, sections 5-8.

Part A:
From Artin, do these problems (given at the end of Chapter 10):
Section 10.5: 2, 4, 6; 10.7: 3, 7, 10 (and relate your answer to the last problem on PS #1).

Part B:
1. Let R be a commutative ring. Show that if R is an integral domain then the characteristic of R is either 0 or prime. Does the converse hold?

2. Let R be a commutative ring and $f(x) \in R[x]$ a polynomial of degree $n > 0$.

a) Show that if R is an integral domain then $f(x)$ has at most n roots in R. [Hint: Use induction to show that if $a_1, \ldots, a_r \in R$ are distinct roots of $f(x)$, then the product $(x - a_1) \cdots (x - a_r)$ divides $f(x)$.]

b) Show by example that the same assertion need not hold if R is not an integral domain. [Hint: Try $R = \mathbb{Z}/8[x]$ and $f(x) = x^2 - c$ for some $c \in \mathbb{Z}/8$.] Where does your proof in part (a) break down in this situation?

3. Which of the following ideals are maximal in the indicated rings? For those that are not, find a maximal ideal containing the given ideal. Explain your assertions. [Caution: One of these is tricky.]
$(x-3) \subset \mathbb{Q}[x]$; $(x-3) \subset \mathbb{Z}[x]$; $(x^2-3) \in \mathbb{R}[x]$; $(x^2+3) \in \mathbb{R}[x]$; $(x-3) \subset \mathbb{C}[x,y]$; $(x^2+1, y-3) \subset \mathbb{R}[x,y]$; $(x^2+1, y-3) \subset \mathbb{C}[x,y]$; $(x^2+1, y^2+1) \subset \mathbb{R}[x,y]$.

Part C:
From Artin, do these problems (at the end of Chapter 10):
Section 10.5: 16; 10.7: 9, 11; Miscellaneous problems: 2.

Also do the following problem:

a) Show that $\sqrt{2}$ is irrational.

b) More generally, show that if $m \in \mathbb{Z}$ and $x^2 - m$ has no root in $\mathbb{Z}$, then $x^2 - m$ has no root in $\mathbb{Q}$.

c) Still more generally, show that if $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}$, and if the polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ has no root in $\mathbb{Z}$, then it has no root in $\mathbb{Q}$.

d) What if, in part (c), the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ (for some integers $a_0, a_1, \ldots, a_n$) is considered instead?