Math 503 Problem Set #7 Due the week of Feb. 26, 2007, in lab.

Reminder: There will be an exam in class on Mon., Feb. 26, on the material covered up through the week before (through Section 12.2 of Artin.)

Read Artin, Chapter 12, sections 1-5.

Part A:
From Artin, do these problems (given at the end of Chapter 12):
Section 12.1: 1, 3, 12 [Note: $V + W$ means $\{v + w \mid v \in V; w \in W\}$]; 12.2: 1, 4; 12.5: 4.

Part B:
1. Which of the following $\mathbb{Z}$-modules are finitely generated? Which are free?
   a) $M = \mathbb{Z}/5 \times \mathbb{Z}/7$
   b) $M = (5) =$ ideal generated by 5
   c) $M = \mathbb{Z}[\sqrt{3}]$
   d) $M = \mathbb{Z}[\pi]$
   e) $M = \mathbb{Z}[1/2]$
   f) $M = \frac{1}{2}\mathbb{Z} = \{\frac{n}{2} \mid n \in \mathbb{Z}\}$

2. Let R be a commutative ring.
   a) If M, N are R-modules, let $\mathrm{Hom}_R(M, N)$ be the set of R-module homomorphisms $M \to N$. Show that $\mathrm{Hom}_R(M, N)$ is an R-module.
   b) Show that if M and N are free R-modules of rank m and n respectively, then $\mathrm{Hom}_R(M, N)$ is free R-module of rank mn. [Hint: Use matrices.]

3. Let R be a commutative ring.
   a) Show that if $\phi : N_1 \to N_2$ is a homomorphism of R-modules, and if M is an R-module, then there is a homomorphism of R-modules $\phi_* : \mathrm{Hom}_R(M, N_1) \to \mathrm{Hom}_R(M, N_2)$ defined by $\phi_*(f) = \phi \circ f$.
   b) Show that if $\psi : M_1 \to M_2$ is a homomorphism of R-modules, and if N is an R-module, then there is a homomorphism of R-modules $\psi^* : \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N)$ defined by $\psi^*(g) = g \circ \psi$.

Part C:
From Artin, do these problems (at the end of Chapter 12):
Section 12.1: 11; 12.4: 6(a); 12.5: 7.

Also do the following problem:
Consider the ideal $I \subset \mathbb{C}[x, y]$ generated by the set $\{y - x, y - x^2, y - x^3, \ldots\}$. Is I finitely generated? If so, find an explicit finite generating set. Also find the zero locus of I in $\mathbb{C}^2$.