Math 503 Problem Set #8 Due the week of March 12, 2007, in lab.

Read Artin, Chapter 12, sections 4-5.

Part A:
From Artin, do these problems (given at the end of Chapter 12):
12.1: 7; 12.2: 2, 5; 12.4: 1(b); 12.5: 1 (first matrix only), 6.

Part B:
1. Let R be a commutative ring. Recall that if M, N are R-modules, then $\mathrm{Hom}_R(M, N)$ denotes the set of R-module homomorphisms $M \to N$, and this is an R-module.

a) Show that if M is an R-module, then $\mathrm{Hom}_R(R, M)$ is isomorphic to M as an R-module. [Hint: Send f to $f(1)$.]

b) If M is an R-module, define its *dual module* by $M^* = \mathrm{Hom}_R(M, R)$. If $R = \mathbb{Z}$, find M^* in each of these cases: $M = \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}/3$, $\mathbb{Q}$. Also, explain the relationship of dual modules to dual vector spaces.

c) Evaluate $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}/n)$ in each of these cases: $n = 2, 3, 4$.

2. Which of the following R-modules are finitely generated? Which are free?
a) $R = \mathbb{Z}[\sqrt{-5}]$, $M = (2, 1 + \sqrt{-5}) \subset R$
b) $R = \mathbb{R}[x]$, $M = \mathbb{R}[x, y]$
c) $R = \mathbb{R}[x]$, $M = \mathbb{R}[x, y]/(y^2 - x)$
d) $R = \mathbb{R}[x, y]$, $M = (x, y) \subset R$
e) $R = \mathbb{R}[x]$, $M = \mathbb{R}[x, 1/x]$.

3. In the situation of Problem Set #7, problem 3, show that if ϕ is injective then so is ϕ_*. Also show that if ψ is surjective then ψ^* is injective.

Part C:
From Artin, do these problems (at the end of Chapter 12):
Section 12.1: 6; 12.2: 3; Miscellaneous problems: 2.

Also do the following problem:

Let d be a square-free integer, let $K = \mathbb{Q}[\sqrt{d}]$, and let R be the ring of integers of K. Show that R is a Noetherian ring.