Math 503 Problem Set #12 Due the week of April 9, 2007, in lab.

Read Artin, Chapter 13, sections 6-9; and Chapter 14, section 1.

Part A:
From Artin, do these problems (given at the end of Chapter 13): Section 13.5: 1; 13.6: 7, 10; 13.8: 1; miscellaneous problems: 2.

From Artin, do these problems (given at the end of Chapter 14): Section 14.1: 3, 6(a,c).

Part B:
1. Let p be a prime number, and let r, s be positive integers. Show that r divides s if and only if $p^r - 1$ divides $p^s - 1$.

2. Let K be a field, and $f(x) \in K[x]$. Assume that K has characteristic 0. Let $n \geq 1$.

a) Let L be a finite field extension of K, and let $\alpha \in L$. Show that α is a root of f with multiplicity exactly n if and only if $0 = f(\alpha) = f'(\alpha) = \cdots = f^{(n-1)}(\alpha) \neq f^{(n)}(\alpha)$.

b) Show that f has a root (in some extension of K) of multiplicity at least n if and only if $(f(x), f'(x), \ldots, f^{(n-1)}(x))$ is a proper ideal of $K[x]$.

c) What if instead K has non-zero characteristic?

3. Let p be a prime number and let $E = \mathbb{Q}[\zeta_p]$, where ζ_p is a primitive pth root of unity. Let $F = \mathbb{Q}$.

a) Find the degree and the Galois group G of the field extension $F \subset E$.

b) Determine if the extension is separable and normal.

c) Find the fixed field of G in E.

d) Is the extension Galois?

Part C:
From Artin, do these problems (at the end of Chapter 13):
Section 13.6: 15; miscellaneous problems: 3 [Hint: Consider the quadratic factors of this polynomial over the splitting field, and then apply miscellaneous problem 2(a) with $\alpha = 2$, $\beta = 3$].

From Artin, do these problems (at the end of Chapter 14):
Section 14.1: 5, 14.

Also do the following problem:

Let p be a prime number, and let F be a field that contains a primitive pth root of unity ζ_p (i.e. $\zeta_p^p = 1$ but $\zeta_p^n \neq 1$ for $0 < n < p$).

a) Show that F does not have characteristic p.

b) Let $a \in F$ be an element that is not a pth power in F, and let $E = F[\sqrt[p]{a}]$. Repeat parts (a)-(d) of problem B3 for this field extension $F \subset E$.