Math 602 Problem Set #1 Due Mon., Sept. 20, 2004

1. Define the *center* of a group G to be $Z = \{g \in G \mid (\forall h \in G)\, gh = hg\}$.
   a) Is Z a subgroup? Is it normal?
   b) Find the center of $C_n, D_n, S_n, A_n, Q, \mathbb{Z}, GL_2(\mathbb{R})$.

2. If H is a subgroup of G, define the *normalizer* of H by $N(H) = \{a \in G \mid aHa^{-1} = H\}$. Is $N(H)$ a subgroup of G? Is H a subgroup of $N(H)$? Is $H \lhd N(H)$? Is $N(H) \lhd G$?

3. a) If H is a subgroup of G and $H \neq G$, we say that H is a *maximal* subgroup if the only subgroups containing H are itself and G. Show that if H is maximal then so is aHa^{-1}, for any $a \in G$.
   b) Define the *Frattini* subgroup Φ of G to be the intersection of the maximal subgroups of G. Show that $\Phi \lhd G$.
   c) Find the Frattini subgroup Φ of D_4, C_4, and Q. In each case, find G/Φ. Conjecture?

4. a) If $x \in G$, define its *centralizer* $Z(x) = \{g \in G \mid xg = gx\}$. Show that $Z(x)$ is a subgroup of G, and that its index $(G : Z(x))$ equals the number of elements in the conjugacy class $\{gxg^{-1} \mid g \in G\}$ of x.
   b) Consider the conjugacy classes in G that have more than one element. Choose one element from each such class, and gather them together as a set S. Show that $\#G = \#Z + \sum_{x \in S} (G : Z(x))$, where Z is the center of G.

5. Let H and K be subgroups of G. If $k \in K$, call the subgroup kHk^{-1} a K-*conjugate* of H. Show that the number of K-conjugates of H is $(K : K \cap N(H))$, where $N(H)$ is the normalizer of H.

6. Define the *commutator* subgroup G' of G to be the subgroup of G generated by the set $C := \{aba^{-1}b^{-1} \mid a, b \in G\}$.
   a) Show $G' \lhd G$.
   b) Show that G/G' is abelian.
   c) Find G' and G/G' if $G = \mathbb{Z}, D_4, S_3, C_2 \times C_3$.
   $*$ d) Is it always the case that $G' = C$ for an arbitrary group? for a finite group?

7. a) Show that $\operatorname{Inn} G \lhd \operatorname{Aut} G$.
   b) Find $\operatorname{Inn} G$ and $\operatorname{Aut} G$ for $G = S_n$, $n \leq 4$. Conjecture?

8. Prove or disprove: A group is abelian if and only if every subgroup is normal.