

1. Let G be the symmetry group of a regular polyhedron P , let v be a vertex of P , and consider the stabilizer $H = \{g \in G \mid g(v) = v\}$ of v . Is H a subgroup of G ? Is it a normal subgroup?
2. Let $\Sigma \subset \mathbb{R}^3$ be the curve given by the two equations $y^2 = 1 + x$, $z^2 = 1 - x$. Find a faithful action of the Klein four group C_2^2 on Σ that preserves the projection of Σ onto the x -axis. Describe the orbits of this action. Find the points of Σ at which the stabilizer is non-trivial, and explicitly find the stabilizer for each of them.
3. Let G be a finite abelian group, and let p_1, \dots, p_r be the prime numbers dividing $\#G$. Let G_{p_i} be the set of elements of G whose orders are powers of p_i . Show that G_{p_i} is a subgroup of G , and that $G \approx G_{p_1} \times \cdots \times G_{p_r}$.
4. If N is a subgroup of G , call N a *characteristic* subgroup if N is invariant under all automorphisms of G ; i.e. if $\phi(N) = N$ for every $\phi \in \text{Aut}(G)$.
 - a) Show that every characteristic subgroup of a group G is normal in G , but that the converse is not in general true.
 - b) Is the commutator subgroup of G (cf. PS#1, problem 6) characteristic?
 - c) Is the Frattini subgroup of G (cf. PS#1, problem 3) characteristic?
 - d) Is the subgroup N , generated by the squares in G and the elements in G of order 17, characteristic?
 - e) Is every subgroup of index 2 characteristic?
 - f) Consider the following sentence: "If M is a _____ subgroup of N , and N is a _____ subgroup of G , then M is a _____ subgroup of G ." Of the eight ways of filling in the blanks with "normal" or "characteristic", find which are true. Prove these, and find counterexamples for the others.
5. Find all normal subgroups of the quaternion group Q . Find the Frattini subgroup Φ , the center Z , and the commutator subgroup Q' . What is Q/F ? Q/Z ? Q/Q' ?
6. If S is a subset of a group G , say that S *generates* G if the subgroup of G generated by S is all of G (i.e. no proper subgroup of G contains S). Call $g \in G$ a *non-generator* of G if whenever a subset $S \subset G$ doesn't generate G , neither does $S \cup \{g\}$. Let G be a finite group.
 - a) Show that the Frattini subgroup $\Phi \subset G$ is the set of non-generators of G .
 - b) Show that if S is a subset of G , and \bar{S} is the image of S under $G \rightarrow G/\Phi$, then S generates G if and only if \bar{S} generates G/Φ .
7. Which of the following groups are isomorphic? $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$; $\mathbb{Z}/2 \times \mathbb{Z}/4$; D_4 ; $\mathbb{Z}/4 \times \mathbb{Z}/2$; $\mathbb{Z}/8$; Q ; $(\mathbb{Z}/32)/N$, where N is the subgroup of $\mathbb{Z}/32$ generated by 8; $(\mathbb{Z}/4 \times \mathbb{Z}/8)/H$, where H is the subgroup of $\mathbb{Z}/4 \times \mathbb{Z}/8$ generated by $(1, 2)$.
8. Fix prime number p and a positive integer n .
 - a) Show that if $A \in GL_n(\mathbb{F}_p)$, then the linear transformation of \mathbb{F}_p^n corresponding to A yields a permutation σ of $\mathbb{F}_p^n - \{0\}$.
 - b) Show that the association $A \mapsto \sigma$ defines an injective group homomorphism $H : GL_n(\mathbb{F}_p) \hookrightarrow S_{p^n - 1}$.
 - c) Find all pairs (p, n) such that H is an isomorphism of $GL_n(\mathbb{F}_p)$ with $S_{p^n - 1}$.