Math 602 Problem Set #4 Due Mon., Oct. 11, 2004

1. Let G be a p-group, and let Φ be its Frattini subgroup.

a) Show that if $g \in G$ then $g^p \in \Phi$. (Hint: If $H \subset G$ is a maximal subgroup, show that $g^p \in H$ by considering its image in G/H.)

b) Deduce that every element of G/Φ has order 1 or p.

c) Conclude that G/Φ is isomorphic to $(\mathbb{Z}/p)^n = \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$ (with n factors) for some $n \geq 0$.

2. If K is a group and S is a subset of K that generates K, we will call S a *minimal generating set* for K if no proper subset of S also generates K.

a) Show that every minimal generating set of $(\mathbb{Z}/p)^n$ has exactly n elements. (Hint: View $(\mathbb{Z}/p)^n$ as a vector space.)

b) Prove or disprove: If G is any finite group, then any two minimal generating sets for G have the same number of elements.

c) Let G be a p-group with Frattini subgroup Φ, so that G/Φ is isomorphic to $(\mathbb{Z}/p)^n$ (as in problem 1(c)). Show that

(i) Every minimal generating set for G has exactly n elements.

(ii) If T is a subset of G with exactly n elements, then T is a minimal generating set for G if and only if its image under $G \twoheadrightarrow G/\Phi$ is a minimal generating set for G/Φ.

(Hint: Use Problem Set 2 #6 and problems 1(c) and 2(a) above.)

(Remark: Part (c) is also called the Burnside Basis Theorem.)

3. For each n, $9 \leq n \leq 16$, answer the following questions:

a) Is every group of order n cyclic?

b) Is every group of order n abelian?

c) Is every abelian group of order n cyclic?

Justify your answers.

4. a) Show that every element of A_5 is conjugate (in A_5) to exactly one of the following five elements:

$$1, (123), (12)(34), (12345), (12354).$$

Determine the number of elements conjugate to each.

b) Deduce that A_5 is simple. [Hint: Show that every normal subgroup is a union of conjugacy classes. Then apply part (a) and Lagrange's Theorem.]

5. Suppose that $N \triangleleft S_5$.

a) Show that if N contains a transposition (a, b) then $N = S_5$. (Hint: The set of transpositions generates S_5.)

b) Show that if $N \cap A_5 = 1$ and $\sigma \in N$, then either $\sigma = 1$ or else σ is a transposition. (Hint: Show that $\sigma^2 = 1$.)

c) Conclude that $N = 1, A_5$, or S_5. [Hint: Apply 5(b) to $N \cap A_5$, and use parts (a) and (b) above.]

6. Show that the three definitions of "solvable" are equivalent: that there is a sequence of subgroups $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ with each G_i/G_{i-1} *abelian* (respectively, *cyclic*, or *cyclic of prime order*).