Math 602 Problem Set #5 Due Mon., Oct. 18, 2004

1. Let $n > 2$. Show that if the dihedral group D_n of order $2n$ is isomorphic to a semi-direct product $C_r \rtimes C_s$, then $r = n$ and $s = 2$.

2. Show that A_4 is isomorphic to a semi-direct product $C_2^2 \rtimes C_3$.

3. Which of the following groups are isomorphic: $C_2 \wr C_2$, Q, D_4, C_2^3?

4. Find all groups of order 66, up to isomorphism. Which are simple? solvable? nilpotent? abelian? cyclic? Which are split extensions (of a non-trivial quotient by a non-trivial subgroup)?

5. a) Show directly that every group of order 56 is solvable. [Hint: How many elements have order 7?]

b) Consider the finite groups whose order is 56 and whose exponent is 14. For each such group, let N_p be the number of Sylow p-subgroups, for $p = 2, 7$.

(i) Do there exist such groups with $N_2 = N_7 = 1$?

(ii) Do there exist such groups with $N_7 = 1$ and $N_2 > 1$?

(iii) Do there exist such groups with $N_2 = 1$ and $N_7 > 1$?

(iv) Do there exist such groups with $N_2 > 1$ and $N_7 > 1$?

6. Find two extensions G of a fixed finite group B by a fixed finite abelian group A such that the two groups G are isomorphic as groups, but such that the two extensions $1 \to A \to G \to B \to 1$ are not isomorphic as extensions of B by A. [Hint: Try $A = C_3^2$ and $B = C_2$.]

7. Show that there is a unique action of C_2 on C_2. With respect to that action, directly compute the groups $C^2(C_2, C_2)$, $Z^2(C_2, C_2)$, $B^2(C_2, C_2)$, $H^2(C_2, C_2)$. In the case of H^2, interpret each element in terms of an extension of C_2 by C_2.

8. Let $0 \to A \xrightarrow{i} G \xrightarrow{\pi} B \to 1$ be a short exact sequence of finite groups, with A abelian (written additively). For each $b \in B$ pick some $g_b \in G$ such that $\pi(g_b) = b$. Define an action α of B on A by $b \cdot a = g_b a g_b^{-1}$. For $b_1, b_2 \in B$, define $f(b_1, b_2) \in A$ by $g_{b_1} g_{b_2} = f(b_1, b_2) g_{b_1 b_2}$.

a) Show that $f \in Z^2_\alpha(B, A)$; i.e. that $f(b_1, b_2) + f(b_1 b_2, b_3) = b_1 \cdot f(b_2, b_3) + f(b_1, b_2 b_3)$. [Hint: Evaluate $g_{b_1} g_{b_2} g_{b_3}$ in two ways.]

b) Show that $(a_1 g_{b_1})(a_2 g_{b_2}) = (a_1 + b_1 \cdot a_2 + f(b_1, b_2)) g_{b_1 b_2} \in G$ for $a_1, a_2 \in A$ and $b_1, b_2 \in B$, giving the multiplication law in G.

c) Suppose that for each $b \in B$ we have another choice $g'_b \in G$ of an element in G with $\pi(g'_b) = b$, and let f' be the analogous element of $Z^2_\alpha(B, A)$. For each $b \in B$ define $e(b) \in A$ by $g'_b = e(b) g_b$. Show that $f'(b_1, b_2) - f(b_1, b_2) = e(b_1) + b_1 \cdot e(b_2) - e(b_1 b_2)$; i.e. f, f' differ by an element of $B^2_\alpha(B, A)$. [Hint: Evaluate $g'_{b_1} g'_{b_2}$ in two ways.]