

1. Which of the following are rings? (Note: To be a ring, it must have a multiplicative identity.) For those which are not, why not? For those which are, are they commutative? integral domains? fields?

- $\mathbb{Q}[e^{2\pi i/15}]$.
- $M_n(\mathbb{Z}[[x]])$.
- $\mathbb{Z} \times \mathbb{R}$, with $(a, b) + (c, d) = (a + c, b + d)$, $(a, b) \cdot (c, d) = (ac, bd)$.
- $\mathbb{R} \times \mathbb{R}$, with $(a, b) + (c, d) = (a + c, b + d)$, $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.
- $\mathbb{Q}((x)) = \{\sum_{i=-n}^{\infty} a_i x^i \mid n \in \mathbb{Z}, a_i \in \mathbb{Q}\}$, the Laurent series in x over \mathbb{Q} .
- $x\mathbb{R}[x]$, the polynomials that are multiples of x .
- $\mathbb{H}_{\mathbb{Z}} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z}\}$, the quaternionic integers, under coordinate-wise addition, and with multiplication as in the quaternion group Q .
- $\{\text{continuous functions } f : \mathbb{R} \rightarrow \mathbb{R}\}$ under addition and multiplication.
- $\{\text{continuous functions } f : \mathbb{R} \rightarrow \mathbb{R}\}$ under addition and composition.
- $\{\text{continuous functions } f : \mathbb{R} \rightarrow \mathbb{R} \text{ with compact support}\}$ (i.e. functions that vanish except in some interval $[-n, n]$), under addition and multiplication.

2. Which of the following are ring homomorphisms? For those which are not, why not? For those which are, what are the kernels and images?

- $f : \mathbb{R}[x] \rightarrow \mathbb{R}$, $f(\sum a_i x^i) = \sum a_i 3^i$ ($a_i \in \mathbb{R}$)
- $f : \mathbb{C} \rightarrow \mathbb{C}$, $f(a + bi) = a - bi$ ($a, b \in \mathbb{R}$)
- $f : \mathbb{C} \rightarrow \mathbb{C}$, $f(a + bi) = a$ ($a, b \in \mathbb{R}$)
- $f : \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{Z}[\sqrt{3}]$, $f(a + b\sqrt{2}) = a + b\sqrt{3}$ ($a, b \in \mathbb{Z}$)
- $f : \mathbb{Z}[x] \rightarrow \mathbb{Q}$, $f(\sum a_i x^i) = \sum a_i / 2^i$ ($a_i \in \mathbb{Z}$)
- $f : \mathbb{Z}[i] \rightarrow \mathbb{Z}/n$, $f(a + bi) = a + 8b$. (Hint: Your answer should depend on n .)

3. Let $\mathbb{H} = \{\text{quaternions } a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$.

- For $\alpha = a + bi + cj + dk \in \mathbb{H}$, define the conjugate $\bar{\alpha} = a - bi - cj - dk$, and define the absolute value $|\alpha| = \sqrt{a^2 + b^2 + c^2 + d^2} \in \mathbb{R}_+$. Show that $|\alpha|^2 = \alpha\bar{\alpha}$ and that $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$. Conclude that $|\alpha\beta| = |\alpha||\beta|$. Also, find all α such that $|\alpha| = 0$.
- Does \mathbb{H} have any (non-zero) zero-divisors? (Hint: Use part (a).)
- Is \mathbb{H} a division ring? a field?

4. Let $R_1 = \mathbb{R}[[x]] =$ the ring of formal power series in x , and let $R_2 = \mathbb{R}\{x\} =$ the ring of convergent power series in x (i.e. power series with a non-zero radius of convergence).

- Show that R_1 and R_2 are integral domains. [Hint: For $f = \sum_{i=n}^{\infty} a_i x^i$ (with $n \geq 0$ and $a_n \neq 0$), define the *order* of f to be the integer $\text{ord}(f) = n$. Define $\text{ord}(0) = \infty$. What is $\text{ord}(f \cdot g)$?]
 - Find all units in R_i .
 - Show that the set of non-units is a principal ideal (f) of R_i , for some $f \in R_i$. Show also that $R_i[1/f]$ is a field, and describe this field explicitly.

5. Show that $M_n(D)$ has no non-trivial two-sided ideals, for any division ring D .

6. Find an example of a non-zero map $f : \mathbb{Z} \rightarrow \mathbb{Z}/n$ (for some n) such that $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{Z}$, but $f(1) \neq 1$ (and so f is not a homomorphism of rings.)