1. Let $I_1, \ldots, I_n \subset R$ be ideals in a commutative ring R.

a) Show that $\prod_{j=1}^n I_j = \bigcap_{j=1}^n I_j$ if the ideals are pairwise relatively prime. Explain this assertion geometrically in the case $R = \mathbb{C}[x, y]$.

b) Let $\phi : R \to \prod_{j=1}^n R/I_j$ be the map obtained by reducing modulo each I_j. Must this map be injective? surjective? an isomorphism? Give examples to show the possibilities, and prove a necessary and sufficient condition for ϕ to be an isomorphism.

2. If $I, J \subset R$ are ideals in a commutative ring, define the *ideal quotient* $(I : J) \subset R$ to be $\{a \in R \,|\, aJ \subset I\}$. Show that this is an ideal. If $R = \mathbb{Z}$, prove that $((m) : (n)) = (m/\gcd(m, n))$.

3. a) Is the Jacobson radical always a radical ideal? Is the nilradical?

b) In each of the following rings, find the Jacobson radical, the nilradical, and the set of units. Also determine if the ring is local. $\mathbb{R}[x, y]$, $\mathbb{R}[[x, y]]$, $\mathbb{R}[x, y]/(y^3)$, $\mathbb{R}[x, y]/(xy)$, $\mathbb{R}[x, y]/(xy, y^3)$, $\mathbb{R}[x][[y]]$, $\mathbb{R}[[y]][x]$.

c) Prove that $\mathbb{R}[x]_{(x)} \subset \mathbb{R}[[x]]$, $\mathbb{R}[x, y]_{(x,y)} \subset \mathbb{R}[[x, y]]$, and $\mathbb{Z}_{(p)} \subset \mathbb{Z}_p$, but that $\mathbb{R}[x, y]_{(y)}$ is not a subring of the completion $\mathbb{R}[x][[y]]$. Explain.

4. If $R \subset S$ are commutative rings and $I \subset R$ is an ideal of R, call $IS \subset S$ the *extension* of I to S. If $J \subset S$ is an ideal of S, call $J \cap R \subset R$ the *contraction* of J to R.

a) Are extensions and contractions always ideals? Are extension and contraction inverse operations?

b) For which prime ideals of $\mathbb{Z}$ is the extension to $\mathbb{Z}[i]$ also prime? For those that are not, which extensions are the product of two distinct prime ideals, and which are the square of a prime ideal of $\mathbb{Z}[i]$? (Of these two cases, the former case is called *split* and the latter case is called *ramified*.)

c) Show that taking contraction induces a surjection from the prime ideals of $\mathbb{Z}[i]$ to the prime ideals of $\mathbb{Z}$. Is it injective?

d) Do your assertions in part (c) hold for an arbitrary extension of integral domains $R \subset S$?

5. Let K be a field.

a) Find all $c \in K$ such that the vectors $(c, 1, 0), (1, c, 1), (0, 1, c) \in K^3$ are linearly independent.

b) Determine whether the vectors $(7, -1, 5), (0, 0, 2) \in K^3$ can be expressed as a linear combination of $v = (3, -1, 2)$ and $w = (1, 1, 1)$; and do so explicitly in each case if possible.

c) Let $T : K^3 \to K^3$ be the homomorphism taking $(1, 0, 0) \mapsto (1, 4, 7)$, $(0, 1, 0) \mapsto (2, 5, 8)$, and $(0, 0, 1) \mapsto (3, 6, 9)$. Describe the kernel and image of T geometrically, find their dimensions, and find a basis for each.

d) Let $S : K^2 \to K^2$ be the homomorphism taking $(a, b) \mapsto (17a - 30b, 9a - 16b)$. Find all diagonal matrices D such that D is the matrix for T with respect to some basis of K^2.

6. a) Let A be a 3×2 matrix and let B be a 2×3 matrix over some field K. Find $\det AB$. Explain.

b) Let A and B be $n \times n$ matrices over K. Show that AB is invertible if and only if both A and B are.