1. a) If A is a square matrix satisfying $A^3 - A^2 + A - I = 0$, find A^{-1} in terms of A.
b) If A is a square matrix satisfying $A^7 = 0$, find $(I - A)^{-1}$ in terms of A.

2. a) Let $V = \{\text{differentiable functions on } \mathbb{R}\}$. Prove that the functions e^x, e^{2x}, e^{3x} are linearly independent in the vector space V. [Hint: If not, differentiate twice.]

b) Let W be the set of solutions to the differential equation $f'' - f = 0$, and let V be the set of solutions to $f''' - f' = 0$. Show that W is a vector subspace of V, find a basis for W, and extend this basis to a basis of V.

3. Let n be an integer, and let $\alpha_1, \ldots, \alpha_{n+1}$ be distinct real numbers. Let $P_n \subset \mathbb{R}[x]$ be the vector space of polynomials of degree $\leq n$. Define $F : P_n \to \mathbb{R}^{n+1}$ by $f \mapsto (f(\alpha_1), \ldots, f(\alpha_{n+1}))$.

a) Show that F is an isomorphism. [Hint: $\dim P_n = ?$ $\ker F = ?$]

b) *Explicitly* find $F^{-1}(e_1), \ldots F^{-1}(e_{n+1})$ (where $e_1, \ldots, e_{n+1}$ are the standard basis vectors in $\mathbb{R}^{n+1}$) in the case $n = 3$, $\alpha_j = j$ $(j = 1, 2, 3, 4)$. [Hint: where does $(x - a)(x - b)(x - c)$ vanish?]

c) Deduce that $F^{-1}(e_1), \ldots F^{-1}(e_{n+1})$ form a basis of P_n. In the case considered in (b), express x as a linear combination of them.

4. a) If V and W are vector spaces over a field K, and if $F : V \to W$ is a homomorphism, let $F^* : W^* \to V^*$ be the map on dual spaces given by $F^*(\phi) = \phi \circ F$. Show that $F \mapsto F^*$ defines a homomorphism $\mathrm{Hom}(V, W) \to \mathrm{Hom}(W^*, V^*)$, satisfying $(F \circ G)^* = G^* \circ F^*$ if $F : V \to W$, $G : U \to V$.

b) Show that the above map $\mathrm{Hom}(V, W) \to \mathrm{Hom}(W^*, V^*)$ is an isomorphism if V and W are finite dimensional.

c) Show that if $0 \to U \to V \to W \to 0$ is exact, then so is $0 \to W^* \to V^* \to U^* \to 0$.

d) What if instead we consider modules over a ring R?

5. For any finite dimensional vector space V with basis $B = \{e_1, \ldots, e_n\}$, and dual basis $B^* = \{\delta_1, \ldots, \delta_n\}$ of V^*, define $\phi_{V,B} : V \to V^*$ by $\sum_1^n a_i e_i \mapsto \sum_1^n a_i \delta_i$, and let $\psi_{V,B} = \phi_{V^*,B^*} \circ \phi_{V,B}$.

a) Show that $\phi_{V,B} : V \to V^*$ is an isomorphism, but that it depends on the choice of B.

b) Show that $\psi_{V,B} : V \to V^{**}$ is an isomorphism which is independent of the choice of B (so we may denote it by ψ_V). For $v \in V$, show that $\psi_V(v)$ is the element of V^{**} taking $f \in V^*$ to $f(v)$.

c) Show that if $F : V \to W$ is a vector space homomorphism with induced homomorphism $F^{**} : V^{**} \to W^{**}$ (notation as in problem 4), then $\psi_W \circ F = F^{**} \circ \psi_V$.

6. If U is a subspace of a vector space V, then the *annihilator* of U is defined to be $\mathrm{Ann}\, U = \{f \in V^* \mid f|_U = 0\}$.

a) Show that $\mathrm{Ann}\, U$ is a subspace of V^*. When can it be all of V^*? When can it be 0?

b) Let V be a finite dimensional vector space and let U, W be subspaces of V. If $V = U \times W$ (internal direct product), show that $V^* = \mathrm{Ann}\, U \times \mathrm{Ann}\, W$.

7. Call $T \in \mathrm{End}(V)$ an *idempotent* if $T^2 = T$. Show that if V is finite dimensional and T is an idempotent, then there are subspaces $X, Y \subset V$ such that $V = X \times Y$, $T|_X = 0$, $T|_Y = \text{identity}$. Deduce that with respect to some basis of V, the idempotent map T is given by a diagonal matrix whose diagonal entries are of the form $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$.