1. Let V, W, Y be finite dimensional vector spaces over K.

a) Show that there are natural isomorphisms $(V \otimes W)^* = V^* \otimes W^* = \mathrm{Hom}(V, W^*) = \mathrm{Hom}(W, V^*)$.

b) Show that there is a natural isomorphism $\mathrm{Hom}(V \otimes W, Y) = \mathrm{Hom}(V, \mathrm{Hom}(W, Y))$.

c) Show that $\mathrm{Hom}(V \otimes W, Y)$ is naturally isomorphic to the vector space of bilinear maps $V \times W \to Y$.

2. Let R be the ring of polynomial functions on the unit sphere $S^2 \subset \mathbb{R}^3$. Thus this ring is given by $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$.

a) Let $P = (0, 0, 1) \in S^2$, and let $R_P = \{\frac{f}{g} \mid f, g \in R;\ g(P) \neq 0\}$. Show directly that R_P is a local ring (i.e. has exactly one maximal ideal I), and find a set of generators for I.

b) Show that $I^2 \subset I$ but that $I^2 \neq I$. Let I/I^2 be the image of I under the ring homomorphism $R_P \to R_P/I^2$. Show that I/I^2 is a 2-dimensional vector space over $\mathbb{R}$. [Hint: Find a basis, using that $z - 1 = \frac{-1}{z+1} \cdot (x^2 + y^2) \in I^2$.]

3. In the situation of problem 2:

a) Let $T \subset \mathbb{R}^3$ be the tangent plane to S^2 at P. Thus $T = \{(x, y, 1) \mid x, y \in \mathbb{R}\}$. Show that T is a 2-dimensional vector space over $\mathbb{R}$, under the addition $(x, y, 1) + (x', y', 1) = (x + x', y + y', 1)$ and scalar multiplication $c(x, y, 1) = (cx, cy, 1)$. What is the 0-vector of T?

b) Let $f \in T^*$, the dual space of T. Show that $f : T \to \mathbb{R}$ extends to a unique linear functional $f' : \mathbb{R}^3 \to \mathbb{R}$. Let $\bar{f} : S^2 \to \mathbb{R}$ be the restriction of f' to S^2. Show that $\bar{f} \in R$, and moreover $\bar{f} \in I$.

c) If $f \in T^*$, let $\phi(f) \in I/I^2$ be the image of $\bar{f} \in I$ under $I \to I/I^2$. Show that $\phi : T^* \to I/I^2$ is an isomorphism of vector spaces.

d) Conclude that T is isomorphic to $(I/I^2)^*$ via ϕ.

Remark. This problem works more generally for any smooth space $S \subset \mathbb{R}^n$ defined by polynomials. Often, geometers turn this problem on its head and *define* the tangent space to be $(I/I^2)^*$. The advantage is that this makes T intrinsic to S, rather than depending on the way that S is embedded in $\mathbb{R}^n$.

4. Let V be a K-vector space and let $0 \to W' \to W \to W'' \to 0$ be an exact sequence of K-vector spaces. Show that the induced sequences $0 \to V \otimes W' \to V \otimes W \to V \otimes W'' \to 0$; $0 \to \mathrm{Hom}(V, W') \to \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W'') \to 0$; and $0 \to \mathrm{Hom}(W'', V) \to \mathrm{Hom}(W, V) \to \mathrm{Hom}(W', V) \to 0$ are also exact. [Hint: Choose a basis for V.]

5. Call $T \in \mathrm{End}\, V$ *nilpotent* if $T^m = 0$ for some $m > 0$. Show that if T is nilpotent, then T has no non-zero eigenvalues.

6. Let V be a vector space. If $T \in \mathrm{End}\, V$ and $W \subset V$, call W a *T-irreducible* subspace if W is T-invariant (i.e. $T(W) \subset W$) and the only T-invariant subspaces of W are 0 and W.

a) Suppose that V is a finite dimensional $\mathbb{C}$-vector space, that $T \in \mathrm{End}\, V$ and its powers form a group of order n under composition, and that W is a T-invariant subspace. Show that W has a T-invariant complement W'. [Hint: Pick an arbitrary complement W'', i.e. $V = W \times W''$. Let $P : V \to W$ be the map $w + w'' \mapsto w$, where $w \in W$, $w'' \in W''$. Define $S : V \to V$ by $v \mapsto \frac{1}{n} \sum_{i=0}^{n-1} T^i P T^{-i}(v)$. Show $S^2 = S$. Then consider $\ker S$ and $\mathrm{im}\, S$.]

b) Under the hypotheses of (a), show that V can be written as the direct product of T-irreducible subspaces.