1. Let V be a vector space and let $G \subset \operatorname{End} V$ be a finite subgroup. Say that $W \subset V$ is *G-invariant* if it is T-invariant for every $T \in G$. Say that $W \subset V$ is *G-irreducible* if W is G-invariant and the only G-invariant subspaces of W are 0 and W.

a) If V is a finite dimensional vector space over $\mathbb{C}$, show that V can be written as the direct product of G-irreducible subspaces.

b) What can you say about the conclusions of parts (a) and of PS 11 problem 6 if the field of scalars is not necessarily $\mathbb{C}$?

2. a) Suppose that $A \in M_n(K)$ is strictly upper triangular (i.e. A is upper triangular, and the diagonal entries are all 0). Show that A is nilpotent, and find the index of nilpotence (i.e. the minimal m such that $A^m = 0$).

b) Show that if S and T are upper triangular, then their *bracket* $[S,T] := ST - TS$ is nilpotent.

c) Let A_0 be the set of upper triangular matrices in $M_n(K)$. Show that A_0 is a *Lie algebra*, in the sense that it is closed under addition, scalar multiplication, and bracket. Also, inductively define A_i by $A_{i+1} = [A_i, A_i] = \langle [S,T] \,|\, S,T \in A_i \rangle$. Show that some $A_r = 0$. (A_0 is thus called *solvable*, in analogy with the fact that a finite group is solvable iff its successive commutators terminate in the trivial group.)

3. Let V be a real inner product space.

a) Show that there is a homomorphism $\phi : V \to V^*$ such that $(\phi(v))(w) = \langle v, w \rangle$ for all $v, w \in V$. Show that ϕ is an isomorphism if V is finite dimensional.

b) If $W \subset V$ is a subspace, what is $\phi(W^\perp)$?

c) If $V = \mathbb{R}^n$ with the usual inner product, describe ϕ explicitly.

4. Prove the Pythagorean Theorem for inner product spaces: If $a \perp b$ (i.e. $\langle a, b \rangle = 0$) then $\|a+b\|^2 = \|a\|^2 + \|b\|^2$. Is the converse true?

5. Let V be a finite dimensional complex inner product space. Show that $T \in \operatorname{End} V$ is Hermitian iff $\langle Tv, v \rangle \in \mathbb{R}$ for all $v \in V$. [Hint: For $\Leftarrow$, show that $\langle (T - T^*)v, v \rangle = 0$.]

6. a) Show that a non-negative matrix $T \in M_r(\mathbb{C})$ has a non-negative n^{th} root, for all $n \in \mathbb{Z}^+$.

b) Show more generally that for such a T we may define, for each real $\alpha \geq 0$, a matrix T^α, such that $T^1 = T$, $(T^\alpha)^\beta = T^{\alpha\beta}$, $T^\alpha T^\beta = T^{\alpha+\beta}$, and T^α varies continuously in α.

c) Regard the positive definite matrices in $M_n(\mathbb{C})$ as a subset $P \subset \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. Show that every element $T \in P$ can be connected to the identity matrix $1 \in P$ by a path in P. [Hint: Part (b).] So P is a connected set.

7. a) Let $A \in M_n(\mathbb{R})$ be a positive definite matrix. Write $x = (x_1, \ldots, x_n)$. Show that

$$\int_{\mathbb{R}^n} e^{-\langle Ax, x \rangle} \, dx_1 \cdots dx_n = \frac{\pi^{n/2}}{\sqrt{\det A}}.$$

(You may use $\int_{-\infty}^{\infty} e^{-t^2} \, dt = \sqrt{\pi}$.) [Hint: A has a square root.]

b) Evaluate $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2+2xy-4y^2-4yz-2z^2} \, dx \, dy \, dz$.