Math 602 Problem Set #6 Due Mon., Oct. 25, 2021

1. a) Show directly that every group of order 56 is solvable. [Hint: How many elements have order 7?]

b) Consider the finite groups whose order is 56 and whose exponent is 14. For each such group, let N_p be the number of Sylow p-subgroups, for $p = 2, 7$.

(i) Do there exist such groups with $N_2 = N_7 = 1$?
(ii) Do there exist such groups with $N_7 = 1$ and $N_2 > 1$?
(iii) Do there exist such groups with $N_2 = 1$ and $N_7 > 1$?
(iv) Do there exist such groups with $N_2 > 1$ and $N_7 > 1$?

2. Find two extensions G of a fixed finite group B by a fixed finite abelian group A such that the two groups G are isomorphic as groups, but such that the two extensions $1 \to A \to G \to B \to 1$ are not isomorphic as extensions of B by A. [Hint: Try $A = C_3^2$ and $B = C_2$.]

3. Show that there is a unique group action of $\mathbb{Z}/2$ on $\mathbb{Z}/2$. With respect to that action, directly compute the groups $C^2(\mathbb{Z}/2, \mathbb{Z}/2)$, $Z^2(\mathbb{Z}/2, \mathbb{Z}/2)$, $B^2(\mathbb{Z}/2, \mathbb{Z}/2)$, $H^2(\mathbb{Z}/2, \mathbb{Z}/2)$. In the case of H^2, interpret each element in terms of an extension of $\mathbb{Z}/2$ by $\mathbb{Z}/2$.

4. With respect to each of the actions of $\mathbb{Z}/2$ on $\mathbb{Z}/3$, compute $H^0(\mathbb{Z}/2, \mathbb{Z}/3)$, $H^1(\mathbb{Z}/2, \mathbb{Z}/3)$, $H^2(\mathbb{Z}/2, \mathbb{Z}/3)$. How does each H^2 relate to group extensions?

5. Let G be a finite group and let p be a prime number. Show that G contains a subgroup F of order prime to p such that for every quotient $E := G/N$ of G of order prime to p, the composition $F \hookrightarrow G \to E$ is surjective. Do this in steps as follows:

i) Let $Q \subseteq G$ be the subgroup generated by all the Sylow p-subgroups of G. Let P be a Sylow p-subgroup of G, and let $G' = N_G(P)$. Show that Q is a normal subgroup of G, and that the quotient map $\pi : G \to H := G/Q$ restricts to a surjection $\pi' : G' \to H$. [Hint: Say $\pi(g) = h$. Must P and gPg^{-1} be conjugate subgroups of Q? Does this yield an element of G' that maps to h?] Show that H is the largest quotient of G of order prime to p.

ii) Deduce that G' (and hence also G) contains a subgroup F having order prime to p such that $\pi(F) = H$, and that F has the desired property. [Hint: With $Q' = N_Q(P)$, consider the exact sequences $1 \to Q' \to G' \to H \to 1$, $1 \to P \to G' \to G'/P \to 1$, and $1 \to Q'/P \to G'/P \to H \to 1$, and apply Schur-Zassenhaus to one of them.]